Complex Fuzzy Sets and Complex Fuzzy Logic an Overview of Theory and Applications

Dan E. Tamir, Naphtali D. Rishe and Abraham Kandel

Abstract Fuzzy Logic, introduced by Zadeh along with his introduction of fuzzy sets, is a continuous multi-valued logic system. Hence, it is a generalization of the classical logic and the classical discrete multi-valued logic (e.g. Łukasiewicz' three/many-valued logic). Throughout the years Zadeh and other researches have introduced extensions to the theory of fuzzy setts and fuzzy logic. Notable extensions include linguistic variables, type-2 fuzzy sets, complex fuzzy numbers, and Z-numbers. Another important extension to the theory, namely the concepts of complex fuzzy logic and complex fuzzy sets, has been investigated by Kandel et al. This extension provides the basis for control and inference systems relating to complex phenomena that cannot be readily formalized via type-1 or type-2 fuzzy sets. Hence, in recent years, several researchers have used the new formalism, often in the context of hybrid neuro-fuzzy systems, to develop advanced complex fuzzy logic-based inference applications. In this chapter we reintroduce the concept of complex fuzzy sets and complex fuzzy logic and survey the current state of complex fuzzy logic, complex fuzzy sets theory, and related applications.

Keywords Fuzzy set theory · Fuzzy class theory · Fuzzy logic · Complex fuzzy sets · Complex fuzzy classes · Complex fuzzy logic · Neuro-fuzzy systems

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1 Introduction

The development of computers and the related attempt to automate human reasoning and inference have posed a challenge to researchers. Humans, and in many cases machines, are not always operating under strict and well defined two-valued logic or discrete multi-valued logic. Their perception of sets and classes is not as crisp as implied by the traditional set and class theory. To capture this perception, L. A. Zadeh has introduced the theory of fuzzy sets and fuzzy logic [1–7]. The seminal paper [1],[1] published by Zadeh in 1965, ignited tremendous interest among a large number of researchers. Following the introduction of the concepts of fuzzy logic and set theory, several researchers, [8–10], have established an axiomatic framework for these concepts.

The five decades that followed Zadhe's pioneering work have produced extensive research work and applications related to control theory [11, 12], artificial intelligence [7, 13–15], inference, and reasoning [16, 17]. In recent years, fuzzy logic has been applied in many areas, including fuzzy neural networks [18], neuro-fuzzy systems and other bio-inspired fuzzy systems [19], clustering [20–22], data mining [13, 23, 24], and software testing [25, 26]. In 1975 Zadeh introduced the concept of linguistic variable and the induced concept of type-2 (type-n) fuzzy sets [3, 27–30]. Other notable extensions to the theory of fuzzy sets and fuzzy logic include complex fuzzy numbers [31], and Z-numbers [32].

Many natural phenomena are complex and cannot be modelled using one-dimensional classes and/or one-dimensional variables. For example, in pattern recognition, objects can be represented by a set of measurements and are regarded as vectors in a multidimensional space. Often, it is not practical to assume that this multidimensional information can be represented via a simple combination of variables and operators on one-dimensional clauses. Specifically, consider a set of values where each value is a member of a fuzzy set. This set, referred to as fuzzy set of type-2, cannot be compactly represented by basic operations on fuzzy sets of type-1 [3, 27–30]. This type of sets however, can be represented via complex classes presented next.

Another important extension to the theory of fuzzy logic and fuzzy sets, namely complex fuzzy logic (CFL) and complex fuzzy sets (CFS), has been developed by Kandel and his coauthors [10, 33–36]. Moses et al. introduced an aggregation of two fuzzy sets into one complex fuzzy set [33]. Next, Ramot et al. introduced the concept of a complex degree of membership represented in polar coordinates, where the amplitude is the degree of membership of an object in a CFS and the role of the phase is to add information which is generally related to spatial or temporal periodicity in the specific fuzzy set defined by the amplitude component. They used this formalism along with the theory of relations to establish the concept of CFL. Finally, Tamir et al. developed an axiomatically-based CFL system and used CFL

[1]The first documented reference by Zadeh to the concepts of Fuzzy Mathematics appeared in a 1962 paper.

to provide a new and general formalism of CFS. These formalisms significantly enhance the expressive power of type-1 and type-2 fuzzy sets [30, 37]. The successive definitions of the theory of CFL and CFS represent an evolution from a relatively naïve and restricted practice to a sound, well founded, practical, and axiomatically-based form. In recent years, several researchers have used the new formalism, often in the context of hybrid neuro-fuzzy systems to develop advanced complex fuzzy logic-based inference applications.

There is a substantial difference between the definitions of complex fuzzy numbers given by J. Buckley [31, 38–41] and the concept of complex fuzzy sets or complex fuzzy logic. Buckley is concerned with generalizing the number theory while the CFL and CFS theories are concerned with the generalization of fuzzy set theory and fuzzy logic [10, 42, 43]. Complex fuzzy numbers have been utilized in several numerical applications [44–46]. Yet, the concept of a complex fuzzy number is different from the concept of complex fuzzy sets or complex fuzzy classes. Recently, Zadeh introduced the concept of *Z-number*s. A *Z*-number, $Z = (A, B)$, is an ordered pair of two fuzzy numbers. In this context A, provides a restriction on a real-valued variable X and B is a restriction on the degree of certainty that X is A [32]. Nevertheless, this concept is used to qualify the reliability of fuzzy quantities rather than to define complex fuzzy sets [10, 36].

The present chapter includes an introduction to the succession of definitions of CFL and CFS, concentrating on the axiomatic-based approach. In addition, the chapter includes a survey the current state of research into complex fuzzy logic, complex fuzzy set theory, and related applications.

The rest of the chapter is organized in the following way: Sect. 2 introduces the axiomatic-based theory of fuzzy set and fuzzy logic. Section 3 surveys the theory of complex fuzzy logic and complex fuzzy sets, concentrating on the axiomatically-based formulation of the theories. Section 4 includes a survey of recent developments in the theory and applications of CFL and CFS. Finally, Sect. 5 presents conclusions and directions for further research.

2 Fuzzy Logic and Set Theory

In 1965, L.A. Zadeh introduced the theory of fuzzy sets, where the degree of membership of an item in a set can get any value in the interval [0, 1] rather than the two values $\{\notin, \in\}$ [1]. Additionally, he introduced the notion of fuzzy logic [1–4]. Fuzzy logic is a continuous (analog) multi-valued extension of classical logic where propositions can get truth values in the interval [0, 1], and are not limited to one of the two values {True, False} (or {0, 1}) [17]. These concepts can be considered as an extension of the multi-valued logic proposed by Łukasiewicz [47]. The introduction of the concepts of fuzzy sets and fuzzy logic was followed by extensive research into fuzzy systems and their applications, related theories, and extensions of the concept [1–4, 6, 13, 17, 19, 22, 26, 48–53]. One direction of research has concentrated on the formulation of an axiomatically-based foundation of fuzzy sets and fuzzy logic [8–10, 54–63]. This is described next.

2.1 Axiomatic Fuzzy Logic

Several researchers presented an axiomatically-based formulation of fuzzy logic and fuzzy set theory [8–10, 55, 56, 58]. In this section we briefly review an axiomatic framework that is founded on the basic fuzzy propositional and predicate logic (BL), along with the fuzzy Łukasiewicz (Ł) and fuzzy product (Π) logical systems [8–10, 55, 56, 58]. We refer to the propositional logic system as ŁΠ and to the first order predicate fuzzy logic system as ŁΠ∀.

Propositional Fuzzy Logic

Several axiom-based logical systems have been investigated [8–10, 55, 56, 58]. Běhounek et al. ([8]) use the ŁΠ/ ŁΠ∀ as the basis for the definition of fuzzy class theory (FCT). Our definition of complex propositional logic presented in Sect. 3 [10, 36], closely follows ŁΠ, the system used by Běhounek et al. For clarity, we reintroduce some of the important notions, notations, and concepts from that paper.

A fuzzy proposition P can get any truth value in the real interval [0, 1], where '0' denotes "False," and '1' denotes "True". Furthermore, the relation $\leq$, over the interval $[0, 1]$ implies a monotonically increasing ordering on the truth values associated with the proposition. A fuzzy interpretation of a proposition P is an assignment of a fuzzy truth value to P. Let P, Q and R denote fuzzy propositions and let $i(R)$ denote the fuzzy interpretation of R. Table 1, includes the basic connectives of ŁΠ. Table 2 includes connectives that can be derived from the basic connectives. The constant 0 is assumed and the constant 1 can be derived from 0 and the basic connectives.

Table 1 Basic ŁΠ connectives

Operation	Interpretation
Ł-Implication	$i(P \to_L Q) = \min(1, 1 - i(P) + i(Q))$
Π-Implication	$i(P \to_{\prod} Q) = \min(1, i(P)/i(Q))$
Π-Conjunction	$i(P \otimes Q) = i(P) \cdot i(Q)$

Table 2 Derived ŁΠ connectives

Operation	Interpretation
Ł-Negation	$i({}'P) = 1 - i(P)$
Π-Delta	$\Delta(i(P)) = 1$ if $i(P) = 1$ else $\Delta(i(P)) = 0$
Equivalence	$i(P \leftrightarrow Q) = i(P \to_L Q) \otimes i(Q \to_L P)$
$P \ominus Q$	$i(P \ominus Q) = \max(0,\ i(P) - i(Q))$

Běhounek et al. use the basic and derived connectives along the truth constants and the following set of axioms [8]:

(1) The Łukasiewicz set of axioms
(2) The product set of axioms
(3) The Łukasiewicz Delta axiom
(4) The Product Delta axiom
(5) The axiom:

$$\mathrm{R} \otimes (P \ominus Q) \leftrightarrow_L (R \otimes P) \ominus (R \otimes Q) \quad (1)$$

The rules of inference are:

(1) Modus ponens
(2) Product necessitation.

Reference [8] includes several theorems that follow from the definition of ŁΠ propositional fuzzy logic. In the next section, we define the ŁΠ first order predicate fuzzy logic (ŁΠ∀).

First Order Predicate Fuzzy Logic

Following the classical logic, the ŁΠ first order predicate fuzzy logic, referred to as ŁΠ∀, extends the ŁΠ propositional fuzzy logic. The primitives include constants, variables, arbitrary-arity functions and arbitrary-arity predicates. Formulae are constructed using (1) the basic connectives defined in Table 1; (2) derived connectives, such as the connectives presented in Table 2; (3) the truth constants; (4) the quantifier ∀ and (5) the identity sign "=". The quantifier ∃ can be used to abbreviate formulae derived from the basic primitives and connectives. A fuzzy interpretation of a proposition $P(x_1, \ldots, x_n)$ over a domain M is a mapping that assigns a fuzzy truth value to each n-tuple of elements of M As in the case of ŁΠ, we closely follow the system used in ref. [8].

Assuming that y can be substituted for x in P and x is not free in Q the following axioms are used:

(1) Instances of the axioms of ŁΠ obtained through substitution
(2) Universal axiom I:

$$(\forall x)P(x) \rightarrow P(y) \quad (2)$$

(3) Universal axiom II:

$$(\forall x)(P \rightarrow_L Q) \rightarrow_L (P \rightarrow_L (\forall x)Q) \quad (3)$$

(4) Identity axiom I:

$$x = x \quad (4)$$

(5) Identity axiom II:

$$(x = y) \to \Delta(P(x) \leftrightarrow P(Y)) \quad (5)$$

Modus ponens, product necessitation, and generalization are used for inference. In the next section, we define propositional and first order predicate CFL.

2.2 Axiomatic Fuzzy Class Theory

The axiomatic fuzzy logic can serve as a basis for establishing an axiomatic FCT. Several variants of FCT exists, most of them use a similar approach and mainly differ in the selection of the logic base. Another difference between various approaches is the selection of class theory axioms [64]. Běhounek et al. present and analyze a few variants of FCT. Ref. [8] presents an ŁΠ∀ based FCT.

3 Complex Fuzzy Logic and Set Theory

The first formalization of complex fuzzy sets and complex fuzzy logic investigated by Kandel and his coauthors [35, 65] is a special case of the formalism presented by Tamir et al. [10]. Hence, in this section only two formalisms for complex fuzzy sets and complex fuzzy logic are considered: (1) the formal definitions provided by Ramot et al. [33], (2) the generalization of these concepts developed by Tamir et al. [10, 36, 43, 66].

3.1 Complex Fuzzy Sets (Ramot et al. [33])

This section reviews the basic concepts and operations of complex fuzzy set as defined by Ramot et al. [34, 67]. According to Ramot et al., a complex fuzzy set S on a universe of discourse U is a set defined by a complex-valued grade of membership function $\mu_s(x)$ [33, 34]:

$$\mu_s(x) = r_s(x)e^{j\omega s(x)} \quad (6)$$

where $j = \sqrt{-1}$. The function $\mu_s(x)$ maps U into the unit disc of the complex plane. This definition utilizes polar representation of complex numbers along with conventional fuzzy set definition; where $r_s(x)$, the amplitude part of the grade of member-ship, is a fuzzy function defined in the interval [0, 1]. On the other hand, $\omega_s(x)$ is a real valued function standing for the phase part of the grade of membership.

In the definition provided by Ramot, the absolute value, or the amplitude part of the membership grade, behaves in the same way as in traditional fuzzy sets. Its value is mapped into the interval [0, 1]. On the other hand, the phase component of the expression is not a fuzzy function; it is a real valued function that can get any real value. Furthermore, the grade of membership is not influenced by the phase. The phase role is to add information which is generally related to spatial or temporal periodicity in the specific fuzzy set defined by the amplitude component. For example, fuzzy information related to solar activity along with crisp information that relates to the date of measurement of the solar activity [33]. Another example where complex fuzzy set has an intuitive appeal comes from the stock market. Intuitively, the periodicity of the stock market along with fuzzy set based estimate of the current values of stocks can be represented by a complex grade of membership such as the one proposed by Ramot. The amplitude conveys the information contained in a fuzzy set such as "strong stock" while the phase conveys a crisp information about the current phase in the presumed stock market cycle.

Following the basic definition of complex-valued grade of membership function Ramot et al. define the basic set operations such as complement, union, and intersection. Each of these operations is defined via a set of theorems [42].

3.2 Complex Fuzzy Logic (Ramot et al. [34])

There are several ways to define fuzzy logic, fuzzy inference, and fuzzy logic system (FLS). One of these ways is to use fuzzy set theory to define fuzzy relations, and then define logical operations, such as implication and negation, as well as inference rules, as special types of relations on fuzzy sets. Alternatively, fuzzy logic can be formalized as a direct generalization of classical logic. Under this "traditional" approach, notions that relate to the syntax and semantics of classical logic, such as propositions, interpretation, and inference are used to define fuzzy logic. Although the relations-based definition can be carefully formalized, it is generally less rigorous than the traditional approach.

Ramot et al. use the first approach [34]. They use the definition of complex fuzzy relations to define complex fuzzy logic via the definition of logical operations. Additionally, Ramot et al. restrict complex fuzzy logic to propositions of the form '*X is A*' , where *X* is a variable that receives values *x* from a universal set *U* and *A* is a complex fuzzy set on *U*. They use this type of propositions to introduce implications of the form '*if X is A then Y is B*' . Finally, they use modus ponens to produce a complex fuzzy inference system. Clearly their approach is limited due to two facts: (1) they rely on complex fuzzy sets and relations to define CFL and (2) their fuzzy inference system is limited to propositions on complex fuzzy sets. These limitations are resolved via the axiomatically-based approach presented in the next section.

3.3 Generalized Complex Fuzzy Logic (Tamir et al. [10])

This section presents the generalized form of complex fuzzy logic investigated by Tamir et al. [10].

Propositional and First Order Predicate Complex Fuzzy Logic

A complex fuzzy proposition P is a composition of two propositions each of which can accept a truth value in the interval $[0, 1]$. In other words, the interpretation of a complex fuzzy proposition is a pair of truth values from the Cartesian interval $[0, 1] \times [0, 1]$. Alternatively, the interpretation can be formulated as a mapping to the unit circle. Formally a fuzzy interpretation of a complex fuzzy proposition P is an assignment of fuzzy truth value of the form $i(p_r) + j \cdot i(p_i)$ or of the form $i(r(p))e^{j\sigma i(\theta(p))}$, where σ is a scaling factor in the interval $(0, 2\pi]$, to P.

For example, consider a proposition of the form "$x \ldots A \ldots B \ldots$," along with the definition of a linguistic variables and constants. Namely, a *linguistic variable* is a variable whose domain of values is comprised of formal or natural language words [3]. Generally, a linguistic variable is related to a fuzzy set such as $\{$*very young male*, *young male*, *old male*, *very old male*$\}$ and can get any value from the set. A linguistic constant has a fixed and unmodified linguistic value, i.e. a single word or phrase from a formal or natural language.

Thus, in a proposition of the form " $x \ldots A \ldots B \ldots$," where A and B are linguistic variables, $i(p_r)$ $(i(r(p)))$ can be assigned to the term A and $i(p_i)$ $(i(\theta(p)))$ can be assigned to term B .

Propositional CFL extends the definition of propositional fuzzy logic and first order predicate CFL extends the notion of first order predicate fuzzy logic. Nevertheless, since propositional CFL is a special case of first order predicate CFL, we only present the formalism for first order predicates CFL here.

Tables 3 and 4 present the basic and derived connectives of ŁΠ∀ CFL. In essence, the connectives are symmetric with respect to the real and imaginary parts of the predicates.

Table 3 Basic ŁΠ∀ CFL connectives

Operation	Interpretation
L-Implication	$i(P \rightarrow_L Q) = \min(1, 1 - i(p_r) + i(q_r)) + j \cdot \min(1, 1 - i(p_i) + i(q_i))$
Π-Implication	$i(P \rightarrow_{\prod} Q) = \min(1, i(p_r)/i(q_r) + j \cdot \min(1,\ i(p_i)/i(q_i))$
Π-Conjunction	$i(P \otimes Q) = i(p_r) \cdot i(q_r)) + j \cdot (i(p_i) \cdot i(q_i))$

Table 4 Derived ŁΠ∀ CFL connectives

Operation	Interpretation
L-Negation	$i(^{'}P) = 1 + j1 - i(P)$
Π-Delta	$\Delta(i(P)) = \text{if}\ \ (\text{i}(P)) = 1 + j1\ \text{else}\ \Delta\ (\text{i}(P)) = 0 + j0$
Equivalence	$i(P \leftrightarrow Q) = i(P_r \ \rightarrow_L Q_r) \otimes i(Q_r \rightarrow_L P_r) + j \cdot i(P_i \rightarrow_L Q_i) \otimes i(Q_i \rightarrow_L P_i)$
$P \ominus Q$	$i(P \ominus Q) = \max(0, i(p_r) - i(q_r)) + j \cdot \max(0, i(p_i) - i(q_i))$

Following classical logic, ŁΠ∀ CFL extends, ŁΠ CFL. The primitives include constants, variables, arbitrary-arity functions and arbitrary-arity predicates. Formulae are constructed using the basic connectives defined in Table 3, derived connectives such as the connectives presented in Table 4, the truth constants, the quantifier ∀ and the identity sign = The quantifier ∃ can be used to abbreviate formulae derived from the basic primitives and connectives. A fuzzy interpretation of a proposition $P(x_1, \ldots, x_n) = P_r(x_1, \ldots, x_n) + j \, . \, P_i(x_1, \ldots, x_m)$ over a domain M is a mapping that assigns a fuzzy truth value to each (n-tuple) × (m-tuple) of elements of M. As in the case of ŁΠ fuzzy logic, we closely follow the system used in ref [8].

The same axioms used for first order predicate fuzzy logic are used for first order predicate complex fuzzy logic; Modus ponens as well as product necessitation, and generalization are the rules of inference.

Complex Fuzzy Propositions and Inference Examples

Consider the following propositions:

1. P(x) ≡ "x is a *destructive hurricane with high surge*"
2. Q(x) ≡ "x is a destructive hurricane with fast moving center"

Let A be the term "*destructive hurricane*." Let B be the term "*high surge*," and let C be the term "*fast moving center*." Hence, P is of the form: "x is a A with B" and Q is of the form "x is A with C" In this case, the terms "*destructve hurricane*," "*high surge*," and "*fast moving center*," are values assigned to the linguistic variables $\{A, B, C\}$. Furthermore, the term "*destructve hurricane*" can get fuzzy truth values (between 0 and 1) or fuzzy linguistic values such as: "*catastriphic*," "*devastating*," and" *disastrous*." Assume that the complex fuzzy interpretation (i.e., the degree of confidence or complex fuzzy truth value) of P is $p_r + jp_i$, while the complex fuzzy interpretation of Q is $q_r + jq_i$. Thus, the truth value of "*x is a devastating hurricane*" is p_R, the truth value of "*x is in a high surge*" is p_i, the truth value of " x is a *catastriphic huricane*" is q_r, and the truth value of "x is a *fast moving center*" is q_i, Suppose that the term "*moderate*" stands for "*non –* destructive" which stands for "*NOT destructive*," the term "*low*" stands for "*NOT high*,", and the term "*slow*" stands for "*NOT fast*." In this context, *NOT* is interpreted as the fuzzy negation operation. Note that this is not the only way to define these linguistic terms and it is used to exemplify the expressive power and the inference power of the logic. Then, the complex fuzzy interpretation of the noted composite propositions is:

(1) $f({}^{\prime}P) = (1 - p_r) + j(1 - p_I)$

That is, ${}^{\prime}P$ denotes the proposition:

"*x is a moderate hurricane with a low surge*." The confidence level in ${}^{\prime}P$ is $(1 - p_r) + j(1 - p_i)$; where the fuzzy truth value of the term "*x is a non – destructive hurricane*," is $(1 - p_r)$ and the fuzzy truth value of the term "low surge," is $(1 - p_i)$.

(2) $'P \rightarrow 'Q = \min(1, q_r - p_r) + j \times \min(1, q_i - p_I)$
Thus, $('P \rightarrow 'Q)$ denotes the proposition: *If* "x is a *moderate* hurricane with a low surge"
THEN x is a *moderate hurricane with low moving center*." The truth values of individual terms, as well as the truth value of $'P \rightarrow 'Q$ are calculated according to Table 1.

(3) $f(P \oplus 'Q) = \max(p_r, 1 - q_r) + j \times \max(p_i, 1 - q_i)$.
That is, $(P \oplus 'Q)$ denotes a proposition such as: "x is a *destructive hurricane with high surge*" OR
"x is a *moderate huricane with slow moving center*" The truth values of individual terms, as well as the truth value of $P \oplus 'Q$ are calculated according to Table 1.

(4) $f('P \otimes Q) = \min(1 - p_r, q_r) + j \times \min(1 - p_i, q_i)$
That is, $('P \otimes Q)$ denotes the proposition "*x is a moderate hurricane with low surge*" AND "*x is a destructive huricane with fast moving center*."
The truth values of individual terms, as well as the truth value of $'P \otimes Q$ are calculated according to Table 1.

Complex Fuzzy Inference Example

Assume that the degree of confidence in the proposition $R = 'P$ defined above is $r_r + jr_i$. Let $S = 'Q$ and assume that the degree of confidence in the fuzzy implication $T = R \rightarrow S$ is $t_r + jt_i$. Then, using Modus ponens

$$R \quad \frac{R \rightarrow S}{S}$$

one can infer S with a degree of confidence $\min(r_r, t_r) + j \times \min(r_i, t_i)$.
In other words if one is using:
"*x is a non − destructive hurricane with a low surge*"
$\underline{IF \text{ "}x \text{ } is \text{ } a \text{ } non - destructive \text{ } hurricane \text{ } with \text{ } a \text{ } low \text{ } surge\text{" } THEN}$
$\underline{\text{"}x \text{ } is \text{ } non - destructive \text{ } huricane \text{ } with \text{ } slow \text{ } moving \text{ } center\text{"}}$
"*x is non − destructive huricane with slow moving center.*"

Hence, using Modus ponens one can infer:
"*x is moderate hurricane with slow moving center*." with a degree of confidence of $\min(r_r, t_r) + j \times \min(r_i, t_i)$.

3.4 *Generalized Complex Fuzzy Class Theory (Tamir et al. [10])*

The axiomatic fuzzy logic can serve as a basis for formal FCT. Similarly, axiomatic based complex fuzzy logic can serve as the basis for formal definition of complex fuzzy classes. In this section we provide a formulation of complex fuzzy class theory (CFCT) that is based on the logic theory presented in Sect. 3.3.

The main components of FCT are:

(1) Variables

(a) Variables denoting objects (potentially complex objects)
(b) Variables denoting crisp sets, i.e. a universe of discourse and its subsets
(c) Variables denoting complex fuzzy classes of order 1
(d) Variables denoting complex fuzzy classes of order n, that is, complex fuzzy classes of complex fuzzy classes of order n-1.

(2) The LΠ∀ CFL system along with its variables, connectives, predicates, and axioms as defined in Sect. 3.3.
(3) Additional predicates

(a) A binary predicate $\in(x,\Gamma)$ denoting membership of objects in complex fuzzy classes and/or in crisp sets

(4) Additional Axioms

(a) Instances of the comprehension schema (further explained below)

$$(\exists\Gamma)\Delta(\forall x)(x\in\Gamma\leftrightarrow P(x)) \tag{7}$$

Where x is a complex fuzzy object, Γ is a complex fuzzy class, and $P()$ is a complex fuzzy predicate.
(b) The axiom of extensionality

$$(\forall x)\Delta(x\in\Gamma\leftrightarrow x\in\Psi)\rightarrow\Gamma=\Psi \tag{8}$$

Where, x is a complex fuzzy object, Γ is a complex fuzzy class, and $P()$ is a complex fuzzy predicate.

Note that a grade of membership is not a part of the above specified terms; yet it can be derived or defined using these terms.

The comprehension schema is used to “construct” classes. It has the basic form of: $(\forall x)(x\in\Gamma\leftrightarrow P(x))$. Intuitively, this schema refers to the class $\mathit{\Gamma}$ of all the objects x that satisfy the predicate $P()$. Instances of this schema have the generic form: $(\exists\Gamma)(\forall x)(x\in\Gamma\leftrightarrow P(x))$. Associated with this schema are comprehension terms of the form: $\in\{\mathrm{x}|P(x)\leftrightarrow P(y)\}$. The Δ operation introduced in Eq. 8 is used to produce precise instances of the extensionality schema and ensure the conservatism of comprehension terms.

Fixing a standard model over the CFCT enables the definition of commonly used terms, set operations, and definitions, as well as proving CFCT theorems. Some of these elements are listed here:

(1) The complex characteristic function $\chi_{x\in\Gamma}\equiv\chi_{\Gamma}$ and the grade of membership function $\mu_{x\in\Gamma}\equiv\mu_{\Gamma}$

(2) Complex class constants, α-cuts, iterated complements, and primitive binary operations, such as union, intersection etc. These operations are constructed using the schema $O_P(\Gamma) \equiv \{x | P(x \in \Gamma)\}$. Table 5 lists some of these elements.
(3) Uniform and supreme relations defined in ref. [8] enable the definition of fuzzy class relations such as inclusion
(4) Theorems, primitive fuzzy class operations, and fuzzy class relations [8].

Following the axiomatically-based definition of grade of membership, Eqs. (9-11) can be used as a basis for the definition of "membership grade based" complement, union, and intersection.

Table 5 Derived primitive class operations

Term	Symbol	P	Comments
Empty complex class	Θ	0	
Universal complex class	Φ	1	
Strict complement	$\backslash\Gamma$	$\sim$	$\sim$ stands for Gödel (G) negation
Complex class intersection	$\cap$	$\oplus$	$\oplus$ stands for a G, Ł, or Π conjunction T-norm
Complex class union	$\cup$	$\vee$	$\vee$ stands for a G, Ł, or Π disjunction

Complex Fuzzy Classes and Connectives Examples

In order to provide a concrete example, we define the following complex fuzzy classes using the comprehension schema. Let the universe of discourse be the set of all the stocks that were available for trading on the opening of the New York stock exchange (NYSE) market on January 5, 2015 along with a set of attributes related to historical price performance of each of these stocks.

Consider the following complex propositions:
P(x) ≡ "x is a *volatile stock* in a *strong portfolio*"
Q(x) ≡ "x is a stock in a *decline trend* in a *strong portfolio*"

Then, the proposition: $(\exists\Gamma)\Delta(\forall x)(x \in \Gamma \leftrightarrow (P(x) \otimes Q(x))$, where x is any member of the universe of discourse, defines a complex fuzzy class Γ that can be "described" as the class of "*volatile stocks in a decline trend in strong portfolios*." On the other hand, the proposition $(\exists\Gamma)\Delta(\forall x)(x \in \Gamma \leftrightarrow ({}' P(x) \vee Q(x))$, where x is any member of the universe of discourse, defines a complex fuzzy class Γ that can be "described" as the class of "*non – volatile stocks in a decline trend in strong portfolios*."

3.5 *Pure Complex Fuzzy Classes*

Often it is useful to define complex fuzzy sets via membership functions rather than through axioms. To this end, Tamir et al. have introduced the concept of pure complex fuzzy sets [42]. This concept is reviewed in this section.

The Cartesian representation of the pure complex grade of membership is given in the following way:

$$\mu(V,x) = \mu_r(V) + j\mu_i(z) \tag{9}$$

Where $\mu_r(V)$ and $\mu_i(z)$, the real and imaginary components of the pure complex fuzzy grade of membership, are real value fuzzy grades of membership. That is, $\mu_r(V)$ and $\mu_i(z)$ can get any value in the interval $[0, 1]$. The polar representation of the pure complex grade of membership is given by:

$$\mu(V,x) = r(V)e^{j\sigma\phi(z)} \tag{10}$$

Where $r(V)$ and $\phi(z)$, the amplitude and phase components of the pure complex fuzzy grade of membership, are real value fuzzy grades of membership. That is, they can get any value in the interval $[0, 1]$. The scaling factor σ is in the interval $(0, 2\pi]$. It is used to control the behavior of the phase within the unit circle according to the specific application. Typical values of σ are $\{1, \frac{\pi}{2}, \pi, 2\pi\}$.

The main difference between pure complex fuzzy grades of membership and the complex fuzzy grade of membership proposed by Ramot et al. [33, 34] is that both components of the membership grade are fuzzy functions that convey information about a fuzzy set.

4 Recent Developments in the Theory and Applications of CFL and CFS

In this section we review recent literature on complex fuzzy logic and complex fuzzy sets. First we review papers that enhance the theoretical basis of CFL/CFS. Next, we outline some of the recent reports on CFL/CFS related applications.

4.1 Advances in the Theoretical Foundations of CFL/CFS

Yager et al. have presented the idea of Pythagorean membership grades and the related idea of Pythagorean fuzzy subsets [68]. They have focused on the negation operation and its relationship to the Pythagorean Theorem. Additionally, they examined the basic set operations for the case of Pythagorean fuzzy subsets. Yager et al. further note that the idea of Pythagorean membership grades can provide an interesting semantics for complex number-valued membership grades used in complex fuzzy sets.

Greenfield et al. ([69]) have compared and contrasted the FCS formalism proposed by Ramot et al. ([33]) as well as the "innovation of pure complex fuzzy sets, proposed by Tamir et al. ([42])" with type-2 fuzzy sets [30, 37]. They have concentrated on the rationales, applications, definitions, and structures of these constructs. In addition, they have compared pure complex fuzzy sets with type-2 fuzzy sets in relation to inference operations. They have concluded that complex fuzzy sets and type-2 fuzzy sets differ in their roles and applications. They have identified similarities between pure complex fuzzy sets and type-2 fuzzy sets; but concluded that type-2 fuzzy sets were isomorphic to pure complex fuzzy sets.

Apolloni et al. propose to define and manage a complex fuzzy set by computing its membership function using a few variables quantized into a few elementary granules and elementary functions connecting the variables [70].

Guosheng et al. have introduced three complex fuzzy reasoning schemes: Principal Axis, Phase Parameters, and Concurrence Reasoning Scheme [49]. They have demonstrated that a variety of conjunction operators and implication operators can be selected to compose the corresponding instances of complex reasoning schemes.

Guangquan et al. have investigated various operation properties of complex fuzzy relations [71]. They have defined a distance measure for evaluating the differences between the grades as well as the phases of two complex fuzzy relations. Furthermore, they have used the distance measure to define δ-equalities of complex fuzzy relations. Finally, they have examined fuzzy inference in the framework of δ-equalities of complex fuzzy relations.

Tamir et al. have proposed a complex fuzzy logic (CFL) system that is based on the extended Post multi-valued logic system (EPS) of order $p > 2$, and have demonstrated its utility for reasoning with fuzzy facts and rules. The advantage of this formalism is that it is discrete. Hence, it better fits real time applications, digital signal processing, and embedded systems that use integer processing units.

4.2 Applications of CFL/CFS

A group of researchers working along with Dick have developed the concept of Adaptive Neuro Fuzzy Complex Inference System (ANCFIS) and explored related applications by integrating complex fuzzy logic into Adaptive Neuro Fuzzy Complex Inference System (ANFIS) [72].

Man et al. have extended the concept of ANFIS and introduced ACNFIS [73]. They have applied ANCFIS in time series forecasting. They compared ACNFIS to three commonly cited time series datasets and demonstrated that ACNFIS was able to accurately model relatively periodic data.

An extension of this work, including synthetic time series and several real-world forecasting problems, is presented by Zhifei et al. [74]. They have found that ANCFIS performs well on these problems and is also very parsimonious. Their work demonstrates the utility of complex fuzzy logic on real-world problems.

Aghakhani et al. have developed an online learning algorithm for ACNFIS and applied it to time series prediction [75]. Their experimental results show that the online technique is comparable to existing results, although slightly inferior to the off-line ANCFIS results.

Yazdanbaksh et al. applied ANCFIS to the problem of short-term forecasts of Photovoltaic power generation [76]. They compared ANFIS and radial basis function networks against ANCFIS. Their experimental results have demonstrated that the ANCFIS based approach was more accurate in predicting power output on a simulated solar cell. Additionally, in a recent paper Yazdanbaksh et al. presented a recommended approach to determining input windows that balances the accuracy and computation time [77].

Another group that is active in exploring CFL/CFS applications is led by Li [14, 78]. Li et al. have proposed a novel complex neuro-fuzzy autoregressive integrated moving average (ARIMA) computing approach and applied it to the problem of time-series forecasting [79]. They have found that their new formalism, referred to as CNFS-ARIMA, has excellent nonlinear mapping capability for time-series forecasting.

Additionally, Li et al. have presented a neuro-fuzzy approach using complex fuzzy sets (CNFS) for the problem of knowledge discovery [80]. They have devised a hybrid learning algorithm to evolve the CNFS for modeling accuracy, combining artificial bee colony algorithm and recursive least squares estimator method. They have tested the CNFS based approach in knowledge discovery through experimentation, and concluded that the proposed approach outperforms comparable approaches.

Another application of CNFS presented by Li et al. is adaptive image noise cancelling [81]. Two cases of image restoration have been used to test the proposed approach and have shown a good restoration quality. Additionally, Li et al. have presented a hybrid learning method that enables efficient and quick CNFS convergence procedure and applied the hybrid learning based CNFS to the problem of function approximation [14, 81]. They have concluded that the CNFS shows much better performance than its traditional neuro-fuzzy counterpart and other compared approaches.

Ma et al. applied complex fuzzy sets to the problem of multiple periodic factor prediction (MPFP) [46]. They have developed a product-sum aggregation operator (PSAO), which is a set of complex fuzzy sets. PSAO has been used to integrate information with uncertainty and periodicity. Next, they have developed a PSAO-based prediction (PSAOP) method to generate solutions for MPFP problems. The experimental results indicate that the proposed PSAOP method effectively handles the uncertainty and periodicity in the information of multiple periodic factors simultaneously and can generate accurate predictions for MPFP problems.

Tamir et al. have considered numerous applications of CFL [24, 36, 43, 66, 82, 83]. They have introduced several soft computing based methods and tools for disaster mitigation [24] and epidemic crises prediction [83]. Additionally, they have demonstrated the potential use of complex fuzzy graphs as well as incremental fuzzy clustering in the context of complex and high order fuzzy logic systems. Additionally, they have developed an axiomatic based framework for discrete complex fuzzy logic and set theory [66].

In [82] Tamir et al. have presented a complex fuzzy logic based inference system used to account for the intricate relations between software engineering constraints such as quality, software features, and development effort. The new model concentrates on the requirements specifications part of the software engineering process. Moreover, the new model significantly improves the expressive power and inference capability of the soft computing component in a soft computing based quantitative software engineering paradigm.

5 Conclusion

We have reviewed the theoretical basis of complex fuzzy logic and complex fuzzy sets and the current state of related applications. We have surveyed the research related to the underlying theory as well as recent applications of the theory in complex fuzzy based algorithms and complex fuzzy inference systems.

The concepts of complex fuzzy logic and complex fuzzy sets have undergone an evolutionary process since they were first introduced [35]. The initial definitions were practical but somewhat naïve and limited [33, 34, 65, 67]. The introduction of axiomatically based approach ([10, 36, 43]) has enabled extending the concepts, maintaining practicality, and providing a solid foundation for further theoretical development. Several applications of the new theories have emerged; based on recent reports this area of applications is gaining momentum.

There are numerous fields where fuzzy concepts interact in intricate ways, which can be effectively captured by the semantics of FCL FCS, pure complex fuzzy classes, and discrete signals. We plan to investigate some of these concepts in the near future. Of particular interest are multimedia signals. Often, these signals are represented using complex functions. On the other hand, due to noise, the processing of such signals might require using complex fuzzy logic. We plan to assess the utility of CFS, CFL, and complex fuzzy sets to the processing of signals in certain noisy situations.

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