Complex Intuitionistic Fuzzy Classes

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***Abstract*—A complex fuzzy class is characterized by a pure complex fuzzy grade of membership. Pure complex fuzzy classes are paramount in providing rich semantics for cases where the fuzzy data is periodic with a fuzzy period. Often, however, the available data is contaminated by noise, opposing expert opinions, ambiguity, and false information. This opens the door for using intuitionistic fuzzy sets theory: representing the false information via a degree of non-membership. Several researchers have identified the benefits of integrating the two concepts of complex fuzzy sets and intuitionistic fuzzy sets. Nevertheless, complex fuzzy sets allow for only one component of the degree of membership to be fuzzy. In this paper, we introduce the concept of complex intuitionistic fuzzy classes, which are characterized by pure complex intuitionistic fuzzy grade of membership. We define the basic terms and operations on complex intuitionistic fuzzy classes and provide a motivating example of relevant application.**

Keywords—Complex fuzzy set, complex fuzzy class, complex intuitionistic fuzzy class.

I. Introduction

Fuzzy Logic Theory (FLT), introduced by Zadeh along with his introduction of the Fuzzy Sets Theory (FST), is a continuous multi-valued logic system [1-6]. Hence, it is a generalization of the classical logic and the classical discrete multi-valued logic [7-9]. Throughout the years Zadeh and other researches have introduced extensions to the theory of fuzzy setts and fuzzy logic. One notable extension is the theory of intuitionistic fuzzy sets [10-14]. Other extensions include linguistic variables, type-2 fuzzy sets, complex fuzzy numbers, and Z-numbers [15-21]. Additionally, significant effort has been made to formalize the theoretical basis of the Fuzzy Set Theory and Fuzzy Logic via well-formed axiomatic-based and classical logic-based approaches [8,9]. On the other hand, numerous applications of FST and FLT evolved throughout the years [22-30]. Many of these applications have been further studied and commercialized [31-37].

Another important extension to FST and FLT, namely the concepts of complex fuzzy logic (CFL) and complex fuzzy sets (CFS), has been investigated by Kandel et al. [38-40]. This extension provides the basis for control and inference systems relating to complex phenomena that cannot be readily formalized via type-1 or type-2 fuzzy sets. Hence, in recent years, several researchers have used the new formalism, often in the context of hybrid neuro-fuzzy systems, to develop advanced complex fuzzy logic-based inference applications [42-59]. Moreover, over the years, the concepts of CFL and CFS has been improved, refined, and formalized by Tamir et al. [60-67]. This formalism is adapted in the present paper.

Intuitionistic FST (IFST) is an important concept that enables addressing cases where the data used to establish the fuzzy degree of membership is contaminated, noisy, or ambiguous [10-14]. For example, making inference based on human facial gestures is hard since it is not always clear whether a specific gesture implies membership in a specific fuzzy set. Nevertheless, there might not be a clear other gesture that implies the logical negation of membership. This and other applications have prompted increased interest in IFST and its applications. We did observe, however, the lack of formal theoretical and axiomatic-based foundation of the theory.

Several researchers have identified the benefits of combining CFS with IFST. Nevertheless, the CFS formalisms used were the early, less rigorous, formalisms of CFS [38-42]. Given the importance of the two fields and the significant potential of the hybrid of the FCL theory and the FCS theory and IFST for improving cost effectiveness of practical systems, we have taken the task of reestablishing the hybrid Intuitionistic Complex Fuzzy Set Theory (ICFST), in a well-founded way.

In this paper we first briefly reintroduce the concept of complex fuzzy sets and intuitionistic fuzzy sets. Next, we formalize the combination of the two paradigms using the latest definition of CFS. Additionally, we provide a motivating example.

The rest of the paper is organized in the following way. Section II provides literature review, summarizing the formalisms of CFST, IFST, and the current work in combining the two fields. Section III presents the new formalism where we use the notion of classes rather than the notion of sets to avoid some of the well know paradoxes concerning set theory. Section IV presents the definition of the basic ICFST operators and Section V contains an example of using the new paradigm. Finally, Section VI includes conclusions and proposals for future research.

II. Literature Review

Fuzzy set theory and fuzzy logic has been discussed in numerous papers and books [1-6]. Additional fundamental and theoretical work includes linguistic variables, type-2 fuzzy sets, complex fuzzy numbers, and Z-numbers [15-21]. Numerous applications of the theoretical work for handling uncertainty via fuzzy logic in the form of inference engines, automatic control system, classification applications, pattern recognition, economic planning, and disaster mitigation are currently at research and commercialization stages [22-37].

In the early work on FST, a fuzzy set has been characterized by a fuzzy grade of *membership* in the following way:

A fuzzy set S, defined on a universe of discourse U, is characterized by a membership function $\mu_S(x): U \rightarrow [0,1]$ that assigns to every element $x \in U$ a grade of membership in S. Hence, the fuzzy set S can be represented as:

$$S = \{(x, \mu_S(x)): x \in U\}. \quad (1)$$

At the same time, there has been significant work related to the definition of the complement of a fuzzy set (S^C). One common definition is the following:

$$S^C = \{(x, 1 - \mu_S(x)): x \in U\} \quad (2)$$

Atanassov has introduced the theory of intuitionistic fuzzy sets (IFS) and intuitionistic fuzzy lattices where an IFS (A) has the form [13-14]:

$$A = \{(x,\ \mu_A(x), v_A(x)): x \in U\};$$
$$0 \le\ \mu_A(x) +\ v_A(x) \le 1 \quad (3)$$

In this case, $v_A(x): U \rightarrow [0,1]$ is the grade of *non-membership* of an element $x \in U$. Clearly, every fuzzy set can be represented as an IFS of the form $S = \{(x, \mu_S(x), 1 - \mu_S(x)): x \in U\}$. Several papers and research projects have utilized the Atanassov form of IFST [10-14]. Additionally, Atanassov have defined the following basic operators [13-14]:

Let:

$$\begin{cases} A = \{(x, \mu_A(x), v_A(x)): x \in U\} \\ B = \{(x, \mu_B(x), v_B(x)): x \in U\} \end{cases} \quad (4)$$

Then:

$$\begin{cases} A^C = \{(x, v_A(x), \mu_A(x)): x \in U\} \\ A \cup B = \{(x, \mu_A(x) \oplus \mu_B(x), v_A(x) \odot v_A(x)): x \in U\} \\ A \cap B = \{(x, \mu_A(x) \odot \mu_B(x), v_A(x) \oplus v_A(x)): x \in U\} \end{cases} \quad (5)$$

Here, $\oplus$ and $\odot$ denote the t-norm and t-conorm operators, respectively.

Several research threads have addressed the subject of providing an axiomatic foundation for FST [7-9,68-79]. Other research efforts have attempted to enhance the capability of FST and FLT to express the rich semantics involved in human inference [68-70].

One thread of related research, the introduction of the concept of complex fuzzy logic and complex fuzzy sets has been initiated by Kandel and coauthors [38-41,60-67]. Moses et al. introduced an aggregation of two fuzzy sets into one complex fuzzy set [38]. Ramot et al. introduced the concept of a complex degree of membership represented in polar coordinates, where the amplitude is the degree of membership of an object in a CFS and the role of the phase is to add information which is generally related to spatial or temporal periodicity in the specific fuzzy set defined by the amplitude component. They used this formalism along with the theory of relations to establish the concept of CFL. Finally, Tamir et al. introduced the concept of pure complex fuzzy sets [60]. Additionally, they developed an axiomatically-based CFL system and used CFL to provide a new and general formalism of CFS [59]. These formalisms significantly enhance the expressive power of type-1 and type-2 fuzzy sets [40-41]. The successive definitions of the theory of CFL and CFS represent an evolution from a relatively naïve and restricted practice to a sound, well founded, practical, and axiomatically-based form. In recent years, several researchers have used the new formalism, often in the context of hybrid neuro-fuzzy systems, to develop advanced complex fuzzy logic-based inference applications.

The following are the basic definitions of pure complex fuzzy classes and their operations according to the formulation of Tamir et al. [59].

A complex fuzzy class Γ, defined on a universe of discourse U, is characterized by a pure complex grade of membership $\mu_\Gamma(V, z) = \mu_r(V) + j\mu_i(z)$, where $\mu_r(V)$ and $\mu_i(z)$, the real and imaginary components of the pure complex fuzzy grade of membership, are real value fuzzy grades of membership. That is, $\mu_r(V)$ and $\mu_i(z)$ can get any value in the interval $[0,1]$. Hence, a pure complex fuzzy class Γ can be represented as a set of ordered triples:

$$\Gamma = \{V, z, \mu_\Gamma(V, z): V \in 2^U, z \in U\}, \quad (6)$$

where 2^U denotes the power-set of U, $\mu_\Gamma(V, z)$ is the degree of membership of z in V and the degree of membership of V in Γ, and $\mu_\Gamma(V,z)$ defines a pure fuzzy class of order 1, that is, a fuzzy set of fuzzy sets. In the rest of this paper, we consider a pure fuzzy class of order 1. The values that $\mu_\Gamma(V,z)$ can receive lie within the unit square or unit circle in the complex plane and are in Cartesian representation of the following two forms: the Cartesian form (7) – as used in this paper -- and the polar form (8).

$$\mu_\Gamma(V,z) = \mu_r(V) + j\mu_i(z)$$
$$\text{or } \mu_\Gamma(V,z) = \mu_i(V) + j\mu_r(z), \quad (7)$$

where $\mu_r(a)$ and $\mu_i(a)$ are real functions with the range of $[0,1]$. In polar representation, where $r(a)$ and $\phi(a)$ are real functions with a range of $[0,1]$ and $\theta \in (0,2\pi]$:

$$\mu_\Gamma(V,z) = r(V).\, e^{j\theta\phi(z)}$$
$$\text{and } \mu_\Gamma(V,z) = r(z).\, e^{j\theta\phi(V)}, \quad (8)$$

Complement of Complex Fuzzy Class

Let Γ be a complex fuzzy class on U and $\mu_\Gamma(V,z) = \mu_{r_\Gamma}(V) + j\mu_{i_\Gamma}(z)$ be its pure complex fuzzy grade of membership. The complement of Γ, denoted Γ^c, is defined by:

$$c(\mu_\Gamma(V,z)) = c(\mu_{r_\Gamma}(V) + j\mu_{i_\Gamma}(z)) = c(\mu_{r_\Gamma}(V)) + jc(\mu_{i_\Gamma}(z)) = (1-\mu_{r_\Gamma}(V)) + j(1-\mu_{i_\Gamma}(z)). \quad (9)$$

Union of Complex Fuzzy Classes

Let Γ_1 and Γ_2 be two complex fuzzy classes on U, and $\mu_{\Gamma_1}(V,z) = \mu_{r_{\Gamma_1}}(V) + j\mu_{i_{\Gamma_1}}(z)$ and $\mu_{\Gamma_2}(T,z) = \mu_{r_{\Gamma_2}}(T) + j\mu_{i_{\Gamma_2}}(z)$ be their pure complex fuzzy grades of membership, respectively. The complex fuzzy class union of Γ_1 and Γ_2, $\Gamma_1 \cup \Gamma_2$, where $\oplus$ denotes a t-conorm operation, is defined by:

$$\mu_{\Gamma_1\cup\Gamma_2}(W,z) = (\mu_{r_{\Gamma_1}}(V)\oplus\mu_{r_{\Gamma_2}}(T)) + j(\mu_{i_{\Gamma_1}}(z)\oplus\mu_{i_{\Gamma_2}}(z)), \quad (10)$$

Intersection of Complex Fuzzy Classes

Let Γ_1 and Γ_2 be two complex fuzzy classes on U, and $\mu_{\Gamma_1}(V,z) = \mu_{r_{\Gamma_1}}(V) + j\mu_{i_{\Gamma_1}}(z)$ and $\mu_{\Gamma_2}(T,z) = \mu_{r_{\Gamma_2}}(T) + j\mu_{i_{\Gamma_2}}(z)$ be their pure complex fuzzy grades of membership, respectively. The complex fuzzy class intersection of Γ_1 and Γ_2, $\Gamma_1 \cap \Gamma_2$, where $\odot$ denote a t-norm operation, is defined by:

$$\mu_{\Gamma_1\cap\Gamma_2}(W,z) = (\mu_{r_{\Gamma_1}}(V)\odot\mu_{r_{\Gamma_2}}(T)) + j(\mu_{i_{\Gamma_1}}(z)\odot\mu_{i_{\Gamma_2}}(z)), \quad (11)$$

Recently, Zadeh introduced the concept of Z-numbers [21]. A $Z-$number, $Z = (A,B)$, is an ordered pair of two fuzzy numbers. In this context, A provides a restriction on a real-valued variable X and B is a restriction on the degree of certainty that X is A [21]. Nevertheless, this concept is used to qualify the reliability of fuzzy quantities rather than to define complex fuzzy sets [59-64].

Complex intuitionistic fuzzy sets can be successfully applied in multi-attribute decision making problems [10-14]. For example, the concept of pure complex fuzzy class cannot handle the possibility of disagreement between experts in a complex situation. Nor can it deal with false information sources which are in opposite positions concerning the actual pure complex fuzzy grade of membership of a specific set. Pure complex fuzzy grade of membership does not provide the negative aspect of information in a situation which might be inherent in the class. It cannot deal with the information of two or more choices, such as "yes," "no," and "I do not know." Moreover, the pure complex fuzzy grade of membership cannot represent the Atanassov's intuitionistic uncertainty in a class.

Alkouri et al. provided a formulation of intuitionistic complex fuzzy set theory [10]. This interpretation, however, uses the Ramot et al. definition of CFS [40]. Hence, the phase terms of the complex membership function and the complex non-membership function are not fuzzy functions. Thus, they cannot convey fuzzy information and cannot be used as a part of the logical operations performed on complex intuitionistic fuzzy classes. Additionally, this representation of complex membership and non-membership functions is restricted to the polar form. Alkouri et al. [10], discuss and study several operations on complex intuitionistic fuzzy sets, such as complement, union, intersection t-norm, and s-norm. Additional operations on complex intuitionistic fuzzy sets can be found in [10-12]. Further, relations on complex intuitionistic fuzzy sets have been discussed in [11-12]. The work reported in [10-13] provides solid explanation, semantics, and insight into the meaning and significance of ICFS. In this sense, the work reported in this paper is an extension of [10-13] that provides better expressive power.

III. Complex Intuitionistic Fuzzy Classes

In this section, we introduce the concept of pure complex intuitionistic fuzzy grade of membership and the interpretation of pure complex intuitionistic fuzzy grade, as well as coordinate transformation of pure complex intuitionistic fuzzy grade. Additionally, we introduce complex intuitionistic fuzzy classes as a generalization of complex fuzzy classes, and basic operations on complex intuitionistic fuzzy classes.

A. Pure complex intuitionistic fuzzy grade of membership

A pure complex intuitionistic fuzzy grade of membership is the combination of pure complex fuzzy grade of membership $\mu_\Gamma(V,z)$ and pure complex fuzzy grade of non-membership $\upsilon_\Gamma(V,z)$. The Cartesian representation of pure complex intuitionistic fuzzy grade of membership has the following form:

$$\mu_\Gamma(V,z) = \mu_r(V) + j\mu_i(z)$$
$$\text{and } \upsilon_r(V,z) = \upsilon_\Gamma(V) + j\upsilon_i(z). \quad (12)$$

Here, $\mu_r(V), \upsilon_r(V)$ and $\mu_i(z), \upsilon_i(z)$ are the real and imaginary components of the pure complex fuzzy membership grade and non-membership grade, respectively. That is, $\mu_r(V), \upsilon_r(V)$ and $\mu_i(z), \upsilon_i(z)$ can get values in the unit interval $[0,1]$. Furthermore, a constrained is placed on $\mu_r(V), \upsilon_r(V)$ and $\mu_i(z), \upsilon_i(z)$. Under this constraint $0 \le \mu_r(V) + \upsilon_r(V) \le 1$ and

$0 \leq \mu_i(z) + \upsilon_i(z) \leq 1$. The polar form of pure complex fuzzy grade of membership is given as follows:

$$\mu_\Gamma(V,z) = r_\mu(V).\, e^{j\alpha\phi(z)}$$
$$and\ \upsilon_\Gamma(V,z) = r_\upsilon(V).\, e^{j\beta\theta(z)}, \quad (13)$$

where $r_\mu(V), r_\upsilon(V), \phi(z)$, and $\theta(z)$, respectively, are the amplitude and phase components of the pure complex fuzzy grade of membership and grade of non-membership. In this case, the amplitudes are real-valued functions and can get any value in the unit interval [0,1] under the constraint that $0 \leq r_\mu(V) + r_\upsilon(V) \leq 1$ and $0 \leq \phi(z) + \theta(z) \leq 1$. The scaling factors α and β lie within the interval $(0,2\pi]$ and control the behavior of the phase terms of the pure complex intuitionistic fuzzy grade in the unit circle depending on the specific applications. Their typical values are $\left\{1, \frac{\pi}{2}, \pi, 2\pi\right\}$. Without loss of generality, in this paper we assume the values $\alpha, \beta = 2\pi$. The described constraints are maintained by the basic ICFS operations.

B. Complex Intuitionistic Fuzzy Classes

Let U be a universe of discourse and let 2^U be the power-set of U. Let $V \in 2^U$ and $z \in U$. Then, a complex intuitionistic fuzzy class Γ is characterized by a pure complex intuitionistic fuzzy grade of membership $(\mu_\Gamma(V,z)\ ,\ \upsilon_\Gamma(V,z))$, that is, Γ is characterized by a pure complex fuzzy grade of membership $\mu_\Gamma(V,z)$ and a pure complex fuzzy grade of non-membership $\upsilon_\Gamma(V,z)$. The complex intuitionistic fuzzy class Γ is represented in the following way:

$$\Gamma = \{< V, z, (\mu_\Gamma(V,z), \upsilon_\Gamma(V,z)) > : V \in 2^U, z \in U\} \quad (14)$$

which can get the Cartesian or polar form of the pure complex intuitionistic fuzzy grade of membership. In this paper, we assume that $\mu_r(V), \upsilon_r(V)$ and $r_\mu(V), r_\upsilon(V)$ denote the Cartesian and polar degree of membership and non-membership of V in Γ and $\mu_i(z), \upsilon_i(z)$ and $\phi(z), \theta(z)$ denote the Cartesian and polar degree of membership and non-membership of z in V, respectively. The values of $\mu_\Gamma(V,z)$ and $\upsilon_\Gamma(V,z)$ can lie within the unit square or unit circle in the complex plane. Hence, a complex intuitionistic fuzzy class is the generalization of complex fuzzy class. It is actually a complex fuzzy class with an additional term of pure complex fuzzy grade of non-membership which conveys the information concerning non-membership or false membership in a complex and complicated situation. The coordinate transformation of Cartesian to polar and polar to Cartesian is performed in a way that is similar to the one given in [59].

IV. Class Theoretic Operations on Complex Intuitionistic Fuzzy Classes

In this section we define the basic operations on complex intuitionistic fuzzy classes: complement, union, and intersection. These operations are constructed using the Cartesian form of pure complex intuitionistic fuzzy grade. We follow a combination of the approach of Tamir et al. [59] and of Atanassov [13-14].

A. Complex Intuitionistic Fuzzy Class Complement

Let $\Gamma = \{< V, z, (\mu_\Gamma(V,z), \upsilon_\Gamma(V,z)) >: V \in 2^U, z \in U\}$ be a complex intuitionistic fuzzy class which is characterized by $\mu_\Gamma(V,z)$ and $\upsilon_\Gamma(V,z)$, where $\mu_r(\alpha), \mu_i(\alpha)$ and $\upsilon_r(\alpha), \upsilon_i(\alpha)$ represent the real and imaginary parts of $\mu_\Gamma(V,z)$ and $\upsilon_\Gamma(V,z)$ respectively. Then complement of Γ is denoted as Γ^c and is defined in the following way:

$$\Gamma^c = \left\{\begin{matrix} < V, z, c(\mu_\Gamma(V,z)) = \\ (c(\mu_\Gamma(V,z)), c(\upsilon_\Gamma(V,z))) >: V \in 2^U, z \in U \end{matrix}\right\}, \quad (15)$$

where $c(\mu_\Gamma(V,z))$ and $c(\upsilon_\Gamma(V,z))$ are defined to be:

$$\left\{\begin{matrix} c(\mu_\Gamma(V,z)) = \upsilon_\Gamma(V,z) \\ c(\upsilon_\Gamma(V,z)) = \mu_\Gamma(V,z) \end{matrix}\right\} \quad (16)$$

Equation 16 demonstrates that the complement function operates on the set of intuitionistic fuzzy sets which forms the complex intuitionistic fuzzy class Γ. It changes the degree of the pure complex fuzzy grade of membership to the degree of pure complex fuzzy grade of non-membership.

Thus, the complex intuitionistic fuzzy class complement of Γ is:

$$\Gamma^c = \left\{\begin{matrix} < V, z, (\upsilon_\Gamma(V,z), \mu_\Gamma(V,z)) >: \\ V \in 2^U, z \in U \end{matrix}\right\} \quad (17)$$

B. Complex Intuitionistic Fuzzy Class Union

Let $\Gamma_1 = \left\{< V, z, (\mu_{\Gamma_1}(V,z), \upsilon_{\Gamma_1}(V,z)) >: V \in 2^U, z \in U\right\}$ and $\Gamma_2 = \left\{< T, z, (\mu_{\Gamma_2}(T,z), \upsilon_{\Gamma_2}(T,z) >: T \in 2^U, z \in U\right\}$ be two complex fuzzy classes, and $(\mu_{\Gamma_1}(V,z), \upsilon_{\Gamma_1}(V,z))$ and $(\mu_{\Gamma_2}(T,z), \upsilon_{\Gamma_2}(T,z)\ )$ their pure complex fuzzy grades of membership and non-membership respectively. Then, the complex fuzzy class union is defined in the following way:

$$\left\{\begin{matrix} \Gamma_1 \cup \Gamma_2 = \\ < W, z, \left(\mu_{\Gamma_1\cup\Gamma_2}(W,z), \upsilon_{\Gamma_1\cup\Gamma_2}(W,z)\right) >: \\ W \in 2^U, z \in U \end{matrix}\right\} \quad (18)$$

where $\mu_{\Gamma_1\cup\Gamma_2}(W,z)$ and $\upsilon_{\Gamma_1\cup\Gamma_2}(W,z)$ are defined in the following way:

$$\left\{\begin{matrix} \mu_{\Gamma_1\cup\Gamma_2}(W,z) = \\ (\mu_{\Gamma_1}(V) \oplus \mu_{\Gamma_2}(T)) + j(\mu_{i_{\Gamma_1}}(z) \oplus \mu_{i_{\Gamma_1}}(z)) \\ \upsilon_{\Gamma_1\cup\Gamma_2}(W,z) = \\ (\upsilon_{\Gamma_1}(V) \odot \upsilon_{\Gamma_2}(T)) + j(\upsilon_{i_{\Gamma_1}}(z) \odot \upsilon_{i_{\Gamma_2}}(z)) \end{matrix}\right\} \quad (19)$$

C. Complex Intuitionistic Fuzzy Class Intersection

We follow the methodology of Tamir et al. [59] and Atanassov [13-14].

Let $\Gamma_1 = \left\{< V, z, (\mu_{\Gamma_1}(V,z), \upsilon_{\Gamma_1}(V,z)) >: V \in 2^U, z \in U\right\}$ and $\Gamma_2 = \left\{< T, z, (\mu_{\Gamma_2}(T,z), \upsilon_{\Gamma_2}(T,z) >: T \in 2^U, z \in U\right\}$ be two complex fuzzy classes, and $(\mu_{\Gamma_1}(V,z), \upsilon_{\Gamma_1}(V,z))$ and $(\mu_{\Gamma_2}(T,z), \upsilon_{\Gamma_2}(T,z)\ )$ their pure complex fuzzy grades of membership and non-membership respectively. Then, the complex fuzzy class intersection is defined in the following way:

$$\left\{\begin{matrix} \Gamma_1 \cap \Gamma_2 = \\ < W, z, \left(\mu_{\Gamma_1 \cap \Gamma_2}(W,z), \upsilon_{\Gamma_1 \cap \Gamma_2}(W,z)\right) >: \\ W \in 2^U, z \in U \end{matrix}\right\} \quad (20)$$

where $\mu_{\Gamma_1 \cap \Gamma_2}(W,z)$ and $\upsilon_{\Gamma_1 \cap \Gamma_2}(W,z)$ are defined in the following way:

$$\left\{\begin{matrix} \mu_{\Gamma_1 \cap \Gamma_2}(W,z) = \\ (\mu_{\Gamma_1}(V) \odot \mu_{\Gamma_2}(T)) + j(\mu_{i_{\Gamma_1}}(z) \odot \mu_{i_{\Gamma_1}}(z)) \\ \upsilon_{\Gamma_1 \cap \Gamma_2}(W,z) = \\ (\upsilon_{\Gamma_1}(V) \oplus \upsilon_{\Gamma_2}(T)) + j(\upsilon_{i_{\Gamma_1}}(z) \oplus \upsilon_{i_{\Gamma_2}}(z)) \end{matrix}\right\} \quad (21)$$

The meaning of $(\mu_{\Gamma_1}(V) \odot \mu_{\Gamma_2}(T)), (\mu_{i_{\Gamma_1}}(z) \odot \mu_{i_{\Gamma_1}}(z))$, and of $(\upsilon_{\Gamma_1}(V) \oplus \upsilon_{\Gamma_2}(T)), (\upsilon_{i_{\Gamma_1}}(z) \oplus \upsilon_{i_{\Gamma_2}}(z)$ is the same meaning as in the traditional intuitionistic fuzzy intersection.

Proposition: Let Γ_1 be a complex intuitionistic fuzzy class on U. Then the following are true:

$$\left\{\begin{matrix} (\Gamma^c)^c = \Gamma, \\ \Gamma_1 \cup \Gamma_1 = \Gamma_1, \\ \Gamma_1 \cap \Gamma_1 = \Gamma_1. \end{matrix}\right\} \quad (22)$$

Proof: These proposition follows straightforward from the definition of complex intuitionistic fuzzy class complement, union, and complex intuitionistic fuzzy class intersection.

V. Examples

A. Numerical Examples

1) Numerical Example of Complex Intuitionistic Fuzzy Class

Let $U = \{1,2,3\}$ be a universe of discourse and let 2^U be the power-set of U. Let $\mu_r, \upsilon_r: 2^U \longrightarrow [0,1]$ and $\mu_i, \upsilon_i: U \longrightarrow [0,1]$ be mappings. Then a complex intuitionistic fuzzy class is:

$$\left\{\begin{matrix} \Gamma_1 = \\ < \{1\}, 1, (0.5 + j0.3, 0.4 + j0.4)\ >, \\ < \{2\}, 2, (0.4 + j0.1, 0.3 + j0.8) >, \\ < \{3\}, 3, (0.6 + j0.4, 0.3 + j0.5) >, \\ < \{1,2\}, 1, (0.8 + j0.1, 0.1 + j0.6) >, \\ < \{1,2\}, 2, (0.7 + j0.3, 0.2 + j0.4) >, \\ < \{1,3\}, 1, (0.1 + j0.4, 0.8 + j0.4 >, \\ < \{1,3\}, 3, (0.6 + j0.2, 0.1 + j0.5) >, \\ < \{2,3\}, 2, (0.5 + j0.5, 0.5 + j0.5) >, \\ < \{2,3\}, 3, (0.4 + j0.5, 0 + j0.6) >, \\ < \{1,2,3\}, 1, (0.1 + j0.9, 0.9 + j0.1) >, \\ < \{1,2,3\}, 2, (0.4 + j0.6, 0.3 + j0.3) >, \\ < \{1,2,3\}, 3, (0.2 + j0.8, 0.2 + j0.1) > \end{matrix}\right\} \quad (23)$$

As additional examples of complex intuitionistic fuzzy classes consider $(\Gamma_2, {\Gamma_1}^c, \Gamma_1 \cup \Gamma_2, \Gamma_1 \cap \Gamma_2)$ defined in Equations 24-27.

$$\left\{\begin{matrix} \Gamma_2 = \\ < \{1\}, 1, (0.6 + j0.2, 0.3 + j0.7) >, \\ < \{2\}, 2, (0.5 + j0.5, 0.4 + j0.3) >, \\ < \{3\}, 3, (0.7 + j0.3, 0.4 + j0.5) >, \\ < \{1,2\}, 1, (0.01 + j0.7, 0.1 + j0.7) >, \\ < \{1,2\}, 2, (0.2 + j0.5, 0.1 + j0.4) >, \\ < \{1,3\}, 1, (0.3 + j0.3, 0.4 + j0.6 >, \\ < \{1,3\}, 3, (0.9 + j0.4, 0.02 + j0.4) >, \\ < \{2,3\}, 2, (0.1 + j0.6, 0.5 + j0.1) >, \\ < \{2,3\}, 3, (0.8 + j0.1, 002 + j0.7) >, \\ < \{1,2,3\}, 1, (0.4 + j0.3, 0.3 + j0.5) >, \\ < \{1,2,3\}, 2, (0.3 + j0.6, 0.6 + j0.3) >, \\ < \{1,2,3\}, 3, (0.2 + j0.5, 0.4 + j0.5) > \end{matrix}\right\} \quad (24)$$

$$\left\{\begin{matrix} {\Gamma_1}^c = \\ < \{1\}, 1, (0.4 + j04, 0.5 + j0.3) >, \\ < \{2\}, 2, (0.3 + j0.8, 0.4 + j0.1) >, \\ < \{3\}, 3, (0.3 + j0.5, 0.6 + j0.4) >, \\ < \{1,2\}, 1, (0.0.1 + j0.6, 0.8 + j0.1) >, \\ < \{1,2\}, 2, (0.2 + j0.4, 0.7 + j0.3) >, \\ < \{1,3\}, 1, (0.8 + j0.4, 0.1 + j0.4) >, \\ < \{1,3\}, 3, (0.1 + j0.5, 0.6 + j0.2) >, \\ < \{2,3\}, 2, (0.5 + j0.5, 0.5 + j0.5) >, \\ < \{2,3\}, 3, (0 + j0.6, 0.4 + j0.5) >, \\ < \{1,2,3\}, 1, (0.9 + j0.1, 0.1 + j0.9) >, \\ < \{1,2,3\}, 2, (0.3 + j0.3, 0.4 + j0.6) >, \\ < \{1,2,3\}, 3, (0.2 + j0.1, 0.2 + j0.8) > \end{matrix}\right\} \quad (25)$$

$$\left\{\begin{matrix} \Gamma_1 \cup \Gamma_2 = \\ < \{1\}, 1, (0.6 + j0.3, 0.3 + j04) >, \\ < \{2\}, 2, (0.5 + j0.5, 0.3 + j0.3) >, \\ < \{3\}, 3, (0.7 + j0.4, 0.3 + j0.5) >, \\ < \{1,2\}, 1, (0.8 + j0.7, 0.01 + j0.6) >, \\ < \{1,2\}, 2, (0.7 + j0.5, 0.1 + j0.4) >, \\ < \{1,3\}, 1, (0.3 + j0.4, 0.4 + j0.4 >, \\ < \{1,3\}, 3, (0.9 + j0.4, 0.02 + j0.4) >, \\ < \{2,3\}, 2, (0.5 + j0.6, 0.5 + j0.1) >, \\ < \{2,3\}, 3, (0.8 + j0.5, 0 + j0.6) >, \\ < \{1,2,3\}, 1, (0.4 + j0.9, 0.3 + j0.1) >, \\ < \{1,2,3\}, 2, (0.4 + j0.6, 0.3 + j0.2) >, \\ < \{1,2,3\}, 3, (0.2 + j0.8, 0.2 + j0.1) > \end{matrix}\right\} \quad (26)$$

$$\left\{\begin{array}{c}\Gamma_1 \cap \Gamma_1 = \\ < \{1\}, 1, (0.5 + j0.2, 0.4 + j0.7) >, \\ < \{2\}, 2, (0.4 + j0.1, 0.4 + j0.8) >, \\ < \{3\}, 3, (0.6 + j0.3, 0.4 + j0.5) >, \\ < \{1,2\}, 1, (0.01 + j0.1, 0.1 + j0.7) >, \\ < \{1,2\}, 2, (0.2 + j0.3, 0.2 + j0.4) >, \\ < \{1,3\}, 1, (0.1 + j0.3, 0.8 + j0.6 >, \\ < \{1,3\}, 3, (0.6 + j0.2, 0.02 + j0.4) >, \\ < \{2,3\}, 2, (0.1 + j0.5, 0.5 + j0.5) >, \\ < \{2,3\}, 3, (0.4 + j0.1, 0.2 + j0.7) >, \\ < \{1,2,3\}, 1, (0.1 + j0.3, 0.9 + j0.5) >, \\ < \{1,2,3\}, 2, (0.3 + j0.6, 0.6 + j0.3) >, \\ < \{1,2,3\}, 3, (0.2 + j0.5, 0.4 + j0.5) > \end{array}\right\} \quad (27)$$

B. Stock and Mutual Funds Performance Data

Consider an example of stock portfolios. Let the universe of discourse be the set of all the stocks that were available for trade on the opening of the New York stock exchange (NYSE) market on January 5, 2016, along with a set of attributes related to historical price performance of each of these stocks. Based on this definition, $U = \{$ set of all the stocks…$\}$, $2^U = \{$any possible portfolio of stocks$\}$. Let W, T, and V be elements of 2^U.

Suppose that on January 5, 2016, a person (Alice) consults with two brokers (Bob and Linda) about a portfolio of stocks holdings. Let $\{B\}$ denote the set of the NYSE stocks in the portfolio recommended by Bob, and let $\{L\}$ denote the set of NYSE stocks in the portfolio proposed by Linda on that day. Furthermore, consider a function (f_1) that associates a number between 0 and 1 with a given portfolio and reflects the fuzzy assertion "is a strong portfolio." For example, this function might reflect the price-to-earnings ratio of the stocks comprising portfolio. In addition, consider two functions (f_2, f_3) that associate a number between 0 and 1 with each stock (i.e., any member z of the universe of discourse U). For example, the function f_2 might be a normalized value of volatility of this stock, while the function f_3 can reflect the fuzzy assertion "the stock z is on a decline route." The functions (f_1, f_2, f_3) can be used to define two pure intuitionistic fuzzy classes of order 1.

Let Γ be the class induced by the pure fuzzy intuitionistic complex grade of membership

$$\left\{\begin{array}{c}\mu_\Gamma(V,x) = \mu_{\Gamma r}(V) + j\mu_{\Gamma i}(z) = f_1(V) + jf_2(z) \\ \upsilon_\Gamma(V,z) = \upsilon_{\Gamma r}(V) + \upsilon_{\Gamma i}(z) = f_1(V) + jf_3(z)\end{array}\right\} \quad (28)$$

Let Ψ be the class induced by the pure fuzzy intuitionistic complex grade of membership:

$$\left\{\begin{array}{c}\mu_\Psi(T,x) = \mu_{\Psi r}(T) + j\mu_{\Psi i}(z) = f_3(T) + jf_2(z) \\ \upsilon_\Psi(T,z) = \upsilon_{\Psi r}(T) + j\upsilon_{\Psi i}(z) = f_3(T) + jf_1(z)\end{array}\right\} \quad (29)$$

An intuitive interpretation of the fuzzy class Γ is that it is induced by the fuzzy assertion "volatile stocks in a strong portfolio that are recommended by the brokers." An intuitive interpretation of the fuzzy intuitionistic class Ψ is that it is induced by the fuzzy assertion "volatile stocks that are in a decline route that are in a strong portfolio." Note that these classes (e.g., "volatile stocks in a strong portfolio") are very different from traditional fuzzy sets that are obtained through basic operators on the fuzzy sets induced by the grades of memberships $f_1(V)$, and $f_2(z)$. Hence, the expressive power of pure complex fuzzy intuitionistic classes is much more comprehensive than the expressive power of traditional fuzzy sets and their respective operations. In addition, it should be noted that the concept of intuitionistic complex fuzzy sets presented by Alkouri cannot be used to represent the information contained in Γ or in Ψ.

The above example can be further expanded. Consider the case where V is restricted to be a member of the crisp set $\{B\}$, and we restrict T to be a member of the crisp set $\{L\}$. Now $< \mu_\Gamma(V,x), \mu_\upsilon(V,x) >$ represents a fuzzy intuitionistic class induced by an assertion such as "volatile stocks in a decline route in the strong portfolio that are recommended by the brokers." Similarly, $< \mu_\Psi(T,x), \upsilon_\Psi(V,z) >$ represents the same assertion; but this time, it estimates the perceptions of the brokers in different way using different functions. That is, it implies the assertion represents the fuzzy intuitionistic class induced by an assertion such as "volatile stocks that are in a decline route in a strong portfolio that are recommended by the brokers."

Let "weak" stand for the fuzzy assertion "not strong" then: $c_1 < \mu_\Psi(T,x), \upsilon_\Psi(V,z) > = < f_1(V) + jf_3(z); f_1(V) + jf_2(z) >$ denotes the class induced by: "volatile stocks that that are in a decline route in the weak portfolio recommended by the brokers."

To further illustrate, let $W \in \{V \cup T\}$ under the above restrictions. That is, W is the set of all the stocks recommended by the two brokers. Using '$\max(\alpha)$' as the t-conorm operator for the union operation, the union of the two classes is obtained by:

$$\left\{\begin{array}{c}\mu_{\Gamma\cup\Psi}(W,z) = \\ (\mu_\Gamma(V) \oplus \mu_\Psi(T)) + j(\mu_{i_\Gamma}(z) \oplus \mu_{i_\Psi}(z)) \\ \upsilon_{\Gamma_1\cup\Psi}(W,z) = \\ (\upsilon_\Gamma(V) \odot \upsilon_\Psi(T)) + j\upsilon_{i_\Psi}(z) \odot \upsilon_{i_\Psi}(z))\end{array}\right\} \quad (30)$$

The induced intuitionistic class contains a set of fuzzy sets. The degree of membership (non-membership) of each fuzzy set in the class is the maximum (minimum) between its degree of membership/non-membership in the set induced by the assertion "Strong volatile stock that is recommended by Bob" and its degree of membership / non-membership in the set induced by the assertion "Strong stock in a decline route that is recommended by Linda." The degree of membership of an individual item (stock) within the sets that comprise the class is the maximum (minimum) between its degree of membership (non-membership) in the set induced by the assertion "the stock z is a volatile stock recommended by the brokers." And the set induced by the assertion "the stock z recommended by the brokers is on a decline route." The "end result" is a class that contains the sets and objects induced by a fuzzy assertion such as: "The set of volatile stocks or stocks on a decline route that are in the strong portfolio proposed by the brokers."

Finally, under similar assumptions to those listed above and using '$\min(\alpha)$' as the t-conorm

$$\begin{Bmatrix} \mu_{\Gamma\cap\Psi}(W,z) = \\ (\mu_{\Gamma}(V)\odot\mu_{\Psi}(T)) + j(\mu_{i_{\Gamma}}(z)\odot\mu_{i_{\Psi}}(z)) \\ \upsilon_{\Gamma_1\cap\Psi}(W,z) = \\ (\upsilon_{\Gamma}(V)\oplus\upsilon_{\Psi}(T)) + j\upsilon_{i_{\Psi}}(z)\oplus\upsilon_{i_{\Psi}}(z)) \end{Bmatrix} \quad (31)$$

The induced intuitionistic class contains a set of fuzzy sets. The degree of membership (non-membership) of each fuzzy set in the class is the minimum (maximum) between its degree of membership/non-membership in the set induced by the assertion "Strong volatile stock that is recommended by Bob" and its degree of membership / non-membership in the set induced by the assertion "Strong stock in a decline route is recommended by Linda." The degree of membership of an individual item (stock) within the sets that comprise the class is the minimum (maximum) between its degree of membership (non-membership) in the set induced by the assertion "the stock z is a volatile stock recommended by the brokers." And the set induced by the assertion "the stock z recommended by the brokers is on a decline route."The "end result" is a class that contains the sets and objects induced by a fuzzy assertion such as: "The set of volatile stocks and stocks on a decline route that are in the strong portfolio proposed by the brokers."

VI. Conclusions and further Research

We have introduced the concept of complex intuitionistic fuzzy classes which are characterized by pure complex intuitionistic fuzzy grade of membership. We have provided a definition of the basic terms and operations on complex intuitionistic fuzzy classes and a motivating example of a relevant application. In the future, we plan to provide axiomatic definitions for intuitionistic fuzzy logic and use this formalism to establish a foundation for an axiomatic-based intuitionistic fuzzy class.

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