Complex Number Representation of Intuitionistic Fuzzy Sets

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Abstract **– Complex numbers can capture compound features and convey multifaceted information; thereby providing means for solving complicated problems. In this paper, we combine the degree of membership and a degree of non-membership of members of Intuitionistic Fuzzy Sets via complex numbers to characterize these fuzzy sets. This approach enables extending several concepts such as classical fuzzy sets, Pythagorean fuzzy sets, and complex fuzzy sets. We discuss complex numbers-based set theoretic operations such as union, intersection, and complement. We define the No-Man-Zone (NMZ) set and establish the relation of NMZ characterization with complex numbers. Further, we introduce athematic complex numbers-based operations of intuitionistic fuzzy sets. We show that the square of the absolute of an intuitionistic fuzzy set becomes a Pythagorean fuzzy set. The polar form of an intuitionistic fuzzy set is reduced to a complex fuzzy set.**

Keywords: Fuzzy Set Theory, Intuitionistic Fuzzy Set Theory

I. INTRODUCTION

Fuzzy sets and fuzzy logic were introduced by Zadeh in 1965 in order to handle uncertainty and ambiguity [42, 43]. A fuzzy set is characterized by a membership degree whose range is the unit interval. Fuzzy logic is a multilevel extension to the classical logic such that proposition can get any value in the unit interval instead of one of the two values ‘True’ or ‘False’. Based on the theory of fuzzy sets, several additional concepts, such as interval valued fuzzy sets [43], type-2 fuzzy sets [27, 15, 43], and intuitionistic fuzzy sets [4, 5], have been developed in order to effectively handle uncertainty. Fuzzy sets and fuzzy logic have applications in signal processing [23, 27, 28], control theory [19, 40, 44,], reasoning [22], and data mining [16]. Additional background on fuzzy sets can be found in [6, 7, 13, 31, 35, 36, 37, 46, 47].

Intuitionistic fuzzy sets were introduced by Atanassove in 1986 as a generalization of fuzzy sets by adding the degree of non-membership into the fuzzy set [4]. Thus, an intuitionistic fuzzy set is characterized by a degree of membership (say μ) and a degree of non-membership (ν). And the sum of μ and ν is restricted to be $\mu + \nu \leq 1$. Philosophically this is a bit problematic since we generally assume that the degree of membership and the degree of non-membership of an element of a fuzzy set sums up to 1. The problem is that the residual $(1 - (\mu + \nu))$ is implying the existence of another fuzzy set. This fuzzy set is called the hesitant zone. We refer to this set as the no man zone (NMZ).

A good example for real life occurrence of intuitionistic fuzzy sets is the case of getting advice from two experts. For example, consider a person that consults with two stock brokers. One expert might advise for “strong buy” on a specific stock and the second might advise for “strong sell.” Both terms are implying membership (and non-membership). The combination constructs an intuitionistic fuzzy set along with a zone of uncertainty or hesitation.

Intuitionistic fuzzy sets often better represent fuzziness. Intuitionistic fuzzy sets have been successfully applied in the fields of modeling imprecision [17], decision making problems [27], pattern recognition [40], economics [20], computational intelligence [14], and medical diagnosis [34]. The strength of these concepts evolves from cases where conflicting information concerning membership taints the ability to determine the actual fuzzy membership of objects.

Complex fuzzy set and logic, which are extensions of fuzzy sets and logic respectively, were first proposed by Ramot et al. [32, 33]. According to their definition, a complex fuzzy set is characterized by a complex grade of membership, which is a combination of a traditional fuzzy degree of membership, referred to as the amplitude term, with the addition of an extra term: the phase term. The range of Ramot complex grade of membership is a unit disk in a complex plane. Ramot at al. discuss several complex fuzzy sets operations, such as complement, union, and intersection [32]. The Ramot type of complex fuzzy set can effectively handle cyclic or periodic fuzzy data. Note, however, that the complex fuzzy set concept introduced by Ramot et al. is different from the one defined by Buckley and Zhang [8-11, 45]. Zhang et al. [48] studied other operations and delta-equalities of complex fuzzy sets and complex fuzzy relations. Dick studied complex fuzzy sets and complex fuzzy logics and proposed several applications [18]. Indeed, the applications of complex fuzzy sets span various fields, such as signal processing [48, 32], physical phenomena [32], and economics [18]. Chen et al. developed a neuro-fuzzy architecture using complex fuzzy sets [12]. Jun et al. successfully applied complex fuzzy sets in multiple periodic factor prediction problems [21]. Additional background on complex fuzzy sets can be found in [23, 25, 26, 29, 30]. Similarly to the case of an intuitionistic fuzzy set, a complex intuitionistic fuzzy set is characterized by a complex grade of membership and complex grade of non-membership [1]. The complex intuitionistic fuzzy sets enable defining the values of membership and non-membership for any object in these complex-valued functions. Set theoretic operations, such as complement, union, and intersection, of complex intuitionistic fuzzy sets have been studied in [1]. Complex intuitionistic fuzzy relations are discussed in [2]. Recently, complex intuitionistic fuzzy sets have been applied in multi-attribute decision-making solutions [2, 3].

The Ramot complex fuzzy sets [32] and complex intuitionistic fuzzy sets as defined in [1] are limited to a polar representation, where only the amplitude terms are fuzzy functions that convey fuzzy information. To overcome this issue, Tamir et al. introduced a new concept of pure complex fuzzy grade of membership, where both the real and imaginary parts are fuzzy functions that convey fuzzy information [38]. A pure complex fuzzy membership grade defines a complex fuzzy class and provides a new interpretation of complex fuzzy grade of membership, which represents a complex fuzzy class along with complex fuzzy class operations. It has several advantages over the concepts introduced by Ramot et al. Furthermore, it is a generalization of the complex fuzzy membership grade that provides semantically rich interpretation of complex fuzzy grade of membership in relation to fuzzy class theory [36]. Moreover, Tamir et al. developed an axiomatic approach to complex fuzzy logic and complex fuzzy class theory [39]. They successfully utilized the complex fuzzy classes in problems emerging from physics to stock markets [38]. Complex fuzzy classes can also be used for inference with type-2 fuzzy sets [39]. Recently, Mumtaz et al. [1] introduced complex intuitionistic fuzzy classes.

In the present paper, we use the concept of complex numbers to combine the intuitionistic fuzzy set degree of membership and the degree of non-membership. We develop the theoretical basis and demonstrate several useful features. Additionally, we provide set theoretic operations, definitions, and properties. The main contributions of using complex number representation are 1) It provide for a compact representation of intuitionistic fuzzy sets, 2) using the properties of complex numbers several properties of and relations between intuitionistic fuzzy sets and other concepts such as Pythagorean and complex fuzzy sets are identified.

The rest of the paper is organized in the following way. In Section II we present a literature review. In Section III we introduce the new representation of intuitionistic fuzzy set via complex numbers. Section IV is dedicated to the athematic theory of intuitionistic fuzzy set via complex numbers. Finally, in Section V, we discuss conclusions and propose further research.

II. BASIC CONCEPTS

In this section we present the basic definitions and related notions which are used in the paper.

Definition 1: Let $\ddot{X}$ be a universe of discourse. A fuzzy set A over $\ddot{X}$ is defined by

$$A = \{(x, \mu_A(x)) : x \in \ddot{X}\},$$

where the membership function of A , $\mu_A(x)$ is defined to be $\{\mu_A(x) : \ddot{X} \to [0,1]\}$. For each $x \in \ddot{X}$, the value $\mu_A(x)$ represents the degree of membership of x in the fuzzy set A.

Definition 2: Let $\ddot{X}$ be a universe of discourse. An intuitionistic fuzzy set A over $\ddot{X}$ is defined by

$$A = \{(x, \mu_A(x), \nu_A(x)) : x \in \ddot{X}\}.$$

For each $x \in \ddot{X}$, the $\mu_A(x)$ represents the degree of membership of x in A and $\nu_A(x)$ represents the degree of non-membership to the intuitionistic fuzzy set A.

Definition 3: A complex fuzzy class Γ, defined on a universe of discourse $\ddot{X}$, is characterized by a pure complex grade of membership $\mu_\Gamma(V, x) = \mu_r(V) + j\mu_i(x)$, where $\mu_r(V)$ and $\mu_i(x)$, the real and imaginary components of the pure complex fuzzy grade of membership, are the real value fuzzy grade of membership [38]. That is, $\mu_r(V)$ and $\mu_i(x)$ can get any value in the interval $[0,1]$. Hence a pure complex fuzzy class Γ can be represented as the set of ordered triples:

$$\Gamma = \{V, x, \mu_\Gamma(V, x) : V \in 2^{\ddot{X}}, x \in \ddot{X}\},$$

where $2^{\ddot{X}}$ denotes the power-set of U; $\mu_\Gamma(V, x)$ is the degree of membership of x in V and the degree of membership of V in Γ.

Definition 4: Let $\ddot{X}$ be a universe of discourse and let $2^{\ddot{X}}$ be the power-set of $\ddot{X}$. Let $V \in 2^{\ddot{X}}$ and $x \in \ddot{X}$. Then, a complex intuitionistic fuzzy class Γ is characterized by a pure complex intuitionistic fuzzy grade of membership $\mu_\Gamma(V, x)$ and $\upsilon_\Gamma(V, x)$, that is, Γ is characterized by a pure complex fuzzy grade of membership $\mu_\Gamma(V, x)$ and a pure complex fuzzy grade of non-membership $\upsilon_\Gamma(V, x)$. The complex intuitionistic fuzzy class Γ is represented in the following way:

$$\Gamma = \{< V, x, \mu_\Gamma(V, x), \upsilon_\Gamma(V, x)) >: V \in 2^{\ddot{X}}, x \in \ddot{X}\}.$$

III. INTUITIONISTIC FUZZY SET VIA COMPLEX NUMBERS

In this section we introduce the complex number representation of intuitionistic fuzzy sets. We also discuss several basic properties thereof. We use the symbol j to denote the imaginary unit satisfying $j^2 = -1$

Definition 5: Let $\ddot{\mathrm{X}}$ be a universe of disclosure and $\ddot{\mathrm{A}}$ be an intuitionistic fuzzy set characterized by the complex function $\ddot{z} = \ddot{\mu} + j\ddot{v}$ where $\ddot{\mu}$ is the degree of membership, $\ddot{v}$ is the degree of non-membership and $\ddot{\mu}, \ddot{v} \in [0,1]$. The intuitionistic fuzzy set $\ddot{\mathrm{A}}$ can be represented as the set of ordered pairs

$$\ddot{\mathrm{A}} = \left\{(\ddot{x}, \ddot{z}\,)|\ddot{x} \in \ddot{\mathrm{X}}\right\}.$$

The range of $\ddot{z}$ is a unit disk in the complex plane. In two dimensional Cartesian coordinate system $\ddot{z}$. Fig. 1 depicts the range of $\ddot{z}$ in two-dimensional Cartesian coordinate system.

Example 1: Let $\ddot{X} = \{\ddot{x}_1, \ddot{x}_2, \ddot{x}_3\}$ be a universe of discourse. Then an intuitionistic fuzzy set $\ddot{A}$ representation via complex numbers can be:

$\ddot{A} = \{(\ddot{x}_1, 0.4 + 0.5j), (\ddot{x}_2, 0.6 + 0.2j), (\ddot{x}_3, 0.3 + 0.2j)\}$.

Definition 6: Let $\ddot{\mathrm{A}}$ be an intuitionistic fuzzy set as defined in Definition 5. That is,

$$\ddot{\mathrm{A}} = \left\{(\ddot{x}, \ddot{z}\,)|\ddot{x} \in \ddot{\mathrm{X}}\right\} = \left\{(\ddot{x}, \ddot{\mu} + j\ddot{v})|\ddot{x} \in \ddot{\mathrm{X}}\right\}.$$

The NMZ of $\ddot{\mathrm{A}}$ can be defined in one of the following ways:

1) $NMZ = \left\{(\ddot{x}, (1 - (\ddot{\mu} + \ddot{v}))|\ddot{x} \in \ddot{X}\right\} =$
$= \left\{\left(\ddot{x}, (1 - (\ddot{\mu} + |j\ddot{v}|)\right)|\ddot{x} \in \ddot{X}\right\}.$

2) $NMZ = \left\{(\ddot{x}, (1 - (\ddot{\mu} + j\ddot{v}))|\ddot{x} \in \ddot{X}\right\}$

Note that according to the first definition, which is the common definition, NMZ is a fuzzy set, whereas according to the second definition, NMZ is an intuitionistic fuzzy set.

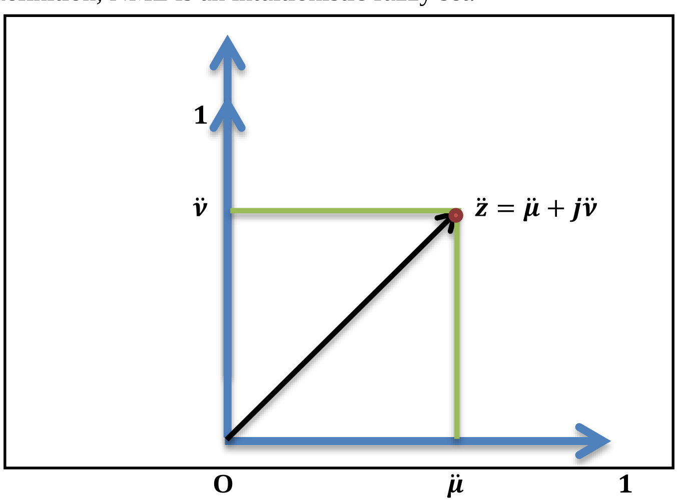

Fig. 1: Complex degree of membership and degree of non-membership in a Cartesian coordinate system

Definition 7: Let $\ddot{z} = \ddot{\mu} + j\ddot{v}$ be a complex number degree of membership function. The real part of $\ddot{z}$ is denoted $Re(\ddot{z})$ and the imaginary part of $\ddot{z}$ is denoted $Im(\ddot{z})$, i.e.:

$$Re(\ddot{z}) = \ddot{\mu}, \text{ and } Im(\ddot{z}) = \ddot{v}.$$

Observation 1: If a complex function $\ddot{z}$ specifies an intuitionistic fuzzy set then the function $Re(\ddot{z})$ specifies a fuzzy set.

By setting $Im(\ddot{z}) = 0$, an intuitionistic fuzzy set via complex number reduces to a fuzzy set. Hence:

Observation 2: The intuitionistic fuzzy sets defined via complex numbers per Definition 5 constitute a generalization of the fuzzy sets.

Definition 8: Let $\ddot{\mathrm{A}}$ and $\ddot{\mathrm{B}}$ be two intuitionistic fuzzy sets which are characterized by two complex functions $\ddot{z}_1 = \ddot{\mu}_1 + j\ddot{v}_1$ and $\ddot{z}_2 = \ddot{\mu}_2 + j\ddot{v}_2$, where $\ddot{\mu}_1, \ddot{\mu}_2$ and $\ddot{v}_1, \ddot{v}_2$ denote the membership and non-membership degrees respectively. Then the union of $\ddot{\mathrm{A}}$ and $\ddot{\mathrm{B}}$, denoted as $\ddot{A} \cup \ddot{B}$, is defined as follows:

$$\ddot{A} \cup \ddot{B} = \{\ddot{z}_1 \coprod \ddot{z}_2 = (\ddot{\mu}_1 \vee \ddot{\mu}_2) + j(\ddot{v}_1 \wedge \ddot{v}_2)\},$$

where $\vee$ and $\wedge$ are the max and min operators.

Example 2: Let $\ddot{X} = \{\ddot{x}_1, \ddot{x}_2, \ddot{x}_3\}$ be a universe of discourse. Let $\ddot{\mathrm{A}}$ and $\ddot{\mathrm{B}}$ be two intuitionistic fuzzy sets defined on $\ddot{X}$ as follows:

$$\ddot{A} = \{(\ddot{x}_1, 0.4 + 0.5j), (\ddot{x}_2, 0.6 + 0.2j), (\ddot{x}_3, 0.3 + 0.2j)\},$$

$$\ddot{B} = \{(\ddot{x}_1, 0.8 + 0.1j), (\ddot{x}_2, 0.5 + 0.5j), (\ddot{x}_3, 0.4 + 0.3j)\}.$$

Then,

$$\ddot{A} \cup \ddot{B} = \{(\ddot{x}_1, 0.8 + 0.1j), (\ddot{x}_2, 0.6 + 0.2j), (\ddot{x}_3, 0.4 + 0.2j)\}.$$

Definition 9: Let $\ddot{\mathrm{A}}$ and $\ddot{\mathrm{B}}$ be two intuitionistic fuzzy sets which are characterized by two complex functions $\ddot{z}_1 = \ddot{\mu}_1 + j\ddot{v}_1$ and $\ddot{z}_2 = \ddot{\mu}_2 + j\ddot{v}_2$ where $\ddot{\mu}_1, \ddot{\mu}_2$ and $\ddot{v}_1, \ddot{v}_2$ are the degrees of membership and non-membership respectively. Then the intersection of $\ddot{\mathrm{A}}$ and $\ddot{\mathrm{B}}$, denoted as $\ddot{A} \cap \ddot{\mathrm{B}}$, is defined as follows:

$$\ddot{A} \cap \ddot{B} = \{\ddot{z}_1 \prod \ddot{z}_2 = (\ddot{\mu}_1 \wedge \ddot{\mu}_2) + j(\ddot{v}_1 \vee \ddot{v}_2)\}$$

where $\vee$ and $\wedge$ are the max and min operators.

Example 3: Let $\ddot{X} = \{\ddot{x}_1, \ddot{x}_2, \ddot{x}_3\}$ be a universe of discourse. Let $\ddot{\mathrm{A}}$ and $\ddot{\mathrm{B}}$ be two intuitionistic fuzzy sets defined on $\ddot{X}$ as follows:

$$\ddot{A} = \{(\ddot{x}_1, 0.4 + 0.5j), (\ddot{x}_2, 0.6 + 0.2j), (\ddot{x}_3, 0.3 + 0.2j)\},$$

$$\ddot{B} = \{(\ddot{x}_1, 0.8 + 0.1j), (\ddot{x}_2, 0.5 + 0.5j), (\ddot{x}_3, 0.4 + 0.3j)\}.$$

Then,

$$\ddot{A} \cap \ddot{B} = \{(\ddot{x}_1, 0.4 + 0.5j), (\ddot{x}_2, 0.5 + 0.5j), (\ddot{x}_3, 0.3 + 0.3j)\}.$$

Definition 10: Let $\ddot{\mathrm{A}}$ be an intuitionistic fuzzy set characterized by a complex function $\ddot{z} = \ddot{\mu} + j\ddot{v}$, where $\ddot{\mu}$ and $\ddot{v}$ are the membership and non-membership degrees respectively. Then the complement of $\ddot{\mathrm{A}}$ is denoted by $\ddot{\mathrm{A}}^c$ and is defined as follows:

$$\ddot{\mathrm{A}}^c = \ddot{z}^c = \ddot{v} + j\ddot{\mu}$$

Example 4: Let $\ddot{X} = \{\ddot{x}_1, \ddot{x}_2, \ddot{x}_3\}$ be a universe of discourse. An intuitionistic fuzzy set $\ddot{A}$ via complex numbers is:

$$\ddot{A} = \{(\ddot{x}_1, 0.1 + 0.9j), (\ddot{x}_2, 0.4 + 0.5j), (\ddot{x}_3, 0.9 + 0.0j)\}.$$

Then,

$$\ddot{A}^c = \{(\ddot{x}_1, 0.9 + 0.1j), (\ddot{x}_2, 0.5 + 0.4j), (\ddot{x}_3, 0.0 + 0.9j)\}.$$

Theorem 1: Let $\ddot{A}$ be an intuitionistic fuzzy set characterized by a complex function $\ddot{z} = \ddot{\mu} + j\ddot{v}$, where $\ddot{\mu}$ and $\ddot{v}$ are the degrees of membership and non-membership respectively, then

$$(\ddot{A}^c)^c = \ddot{A}.$$

Proof: This follows directly from Definition 10. ■

Definition 11: Let $\ddot{A}$ and $\ddot{B}$ be two intuitionistic fuzzy sets characterized by two complex functions $\ddot{z}_1 = \ddot{\mu}_1 + j\ddot{v}_1$ and $\ddot{z}_2 = \ddot{\mu}_2 + j\ddot{v}_2$, where $\ddot{\mu}_1, \ddot{\mu}_2$ and $\ddot{v}_1, \ddot{v}_2$ are the membership and non-membership degrees respectively. Then $\ddot{A}$ is a subset of $\ddot{B}$, denoted as $\ddot{A} \subseteq \ddot{B}$, iff $\ddot{z}_1 \leq \ddot{z}_2$, i.e. $\forall \ddot{x} \in \ddot{X}$: $(\ddot{\mu}_1(x) \leq \ddot{\mu}_2(x))$ & $(\ddot{v}_1(x) \geq \ddot{v}_2(x))$.

Proposition 1: Let $\ddot{A}, \ddot{B}$ and $\ddot{C}$ be three intuitionistic fuzzy sets. Then

1. $\ddot{A} \cup \ddot{B} = \ddot{B} \cup \ddot{A}$,
2. $\ddot{A} \cap \ddot{B} = \ddot{B} \cap \ddot{A}$,
3. $(\ddot{A} \cup \ddot{B}) \cup \ddot{C} = \ddot{A} \cup (\ddot{B} \cup \ddot{C})$,
4. $(\ddot{A} \cup \ddot{B}) \cap \ddot{C} = (\ddot{A} \cap \ddot{B}) \cup (\ddot{B} \cap \ddot{C})$, and
5. $(\ddot{A} \cap \ddot{B}) \cup \ddot{C} = (\ddot{A} \cup \ddot{B}) \cap (\ddot{B} \cup \ddot{C})$.

Proof: The proof is straightforward. ■

IV. ELEMENTARY OPERATIONS AND PROPERTIES

In this section, we study properties of elementary operations of intuitionistic fuzzy sets using operations such as conjugate, addition, subtraction, multiplication, and division.

Definition 12: Let $\ddot{z} = \ddot{\mu} + j\ddot{v}$ be a complex function where the degree of membership is $\ddot{\mu}$ and the degree of non-membership is $\ddot{v}$. Then the conjugate of $\ddot{z}$, denoted by $\bar{\ddot{z}}$, is defined to be: $\bar{\ddot{z}} = \ddot{\mu} - j\ddot{v}$. This is depicted in **Fig. 2.**

The defined conjugate has no intuitive or physical meaning. Nevertheless, it enables identifying additional features of the complex number representation of intuitionistic fuzzy sets.

Proposition 2: Let $\ddot{z} = \ddot{\mu} + j\ddot{v}$ be a complex function where the membership degree is $\ddot{\mu}$ and the degree of non-membership is $\ddot{v}$. Its conjugate $\bar{\ddot{z}} = \ddot{\mu} - j\ddot{v}$, satisfies:

1. $\ddot{\mu} = \frac{1}{2}(\ddot{z} + \bar{\ddot{z}})$,
2. $j\ddot{v} = \frac{1}{2}(\ddot{z} - \bar{\ddot{z}})$.

Proof: Straightforward. ■

Proposition 3: Let $\ddot{z} = \ddot{\mu} + j\ddot{v}$ be a complex function where the degree of membership is $\ddot{\mu}$ and the degree of non-membership is $\ddot{v}$. Then, $\ddot{z} = \ddot{\mu}$ if and only if $\ddot{z} = \bar{\ddot{z}}$.

Proof: Suppose that $\ddot{z} = \ddot{\mu}$. Then by taking the conjugate of both sides, we have

$$\bar{\ddot{z}} = \bar{\ddot{\mu}} = \ddot{\mu} = \ddot{z}.$$

Conversely, suppose that $\ddot{z} = \bar{\ddot{z}}$. Then

$$\ddot{\mu} + j\ddot{v} = \ddot{\mu} - j\ddot{v}, \text{ (proposition 2 item 1)}$$
$$\ddot{v} = -\ddot{v},$$
$$2\ddot{v} = 0 \Longrightarrow \ddot{v} = 0.$$

Therefore by substituting $\ddot{v} = 0$ in (1), we get $\ddot{z} = \ddot{\mu}$. ■

Theorem 2: Let $\ddot{A}$ be an intuitionistic fuzzy set which is characterized by a complex function $\ddot{z} = \ddot{\mu} + j\ddot{v}$, where $\ddot{\mu}$ and $\ddot{v}$ are the degrees of membership and non-membership respectively. Then $\ddot{A}$ is a fuzzy set if and only if $\ddot{z} = \bar{\ddot{z}}$.

Proof: This follows from Proposition 3. ■

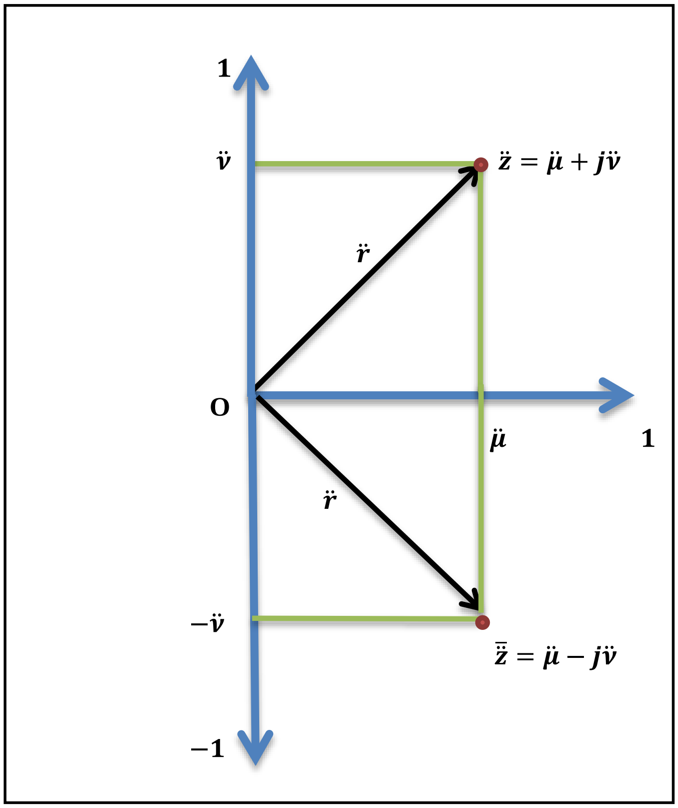

Fig. 2: Conjugate of complex number representation of intuitionistic fuzzy sets

Definition 13: Let $\ddot{z}_1 = \ddot{\mu}_1 + j\ddot{v}_1$ and $\ddot{z}_2 = \ddot{\mu}_2 + j\ddot{v}_2$ be two complex functions where $\ddot{\mu}_1, \ddot{\mu}_2$ and $\ddot{v}_1, \ddot{v}_2$ are the degrees of membership and non-membership respectively, then

$$\ddot{z}_1 + \ddot{z}_2 = (\ddot{\mu}_1 + j\ddot{v}_1) + (\ddot{\mu}_2 + j\ddot{v}_2) = (\ddot{\mu}_1 + \ddot{\mu}_2) + j(\ddot{v}_1 + \ddot{v}_2),$$

$$\ddot{z}_1 - \ddot{z}_2 = (\ddot{\mu}_1 + j\ddot{v}_1) - (\ddot{\mu}_2 + j\ddot{v}_2) = (\ddot{\mu}_1 - \ddot{\mu}_2) + j(\ddot{v}_1 - \ddot{v}_2).$$

Definition 14: Let $\ddot{z}_1 = \ddot{\mu}_1 + j\ddot{v}_1$ and $\ddot{z}_2 = \ddot{\mu}_2 + j\ddot{v}_2$ be two complex functions where $\ddot{\mu}_1, \ddot{\mu}_2$ and $\ddot{v}_1, \ddot{v}_2$ are the membership and non-membership degrees respectively. Then

$$\ddot{z}_1\ddot{z}_2 = (\ddot{\mu}_1 + j\ddot{v}_1)(\ddot{\mu}_2 + j\ddot{v}_2) = (\ddot{\mu}_1\ddot{\mu}_2 - \ddot{v}_1\ddot{v}_2) + j(\ddot{v}_1\ddot{\mu}_2 + \ddot{\mu}_1\ddot{v}_2),$$

$$\frac{\ddot{z}_1}{\ddot{z}_2} = \frac{\ddot{\mu}_1 + j\ddot{v}_1}{\ddot{\mu}_2 + j\ddot{v}_2} = \left(\frac{\ddot{\mu}_1\ddot{\mu}_2 + \ddot{v}_1\ddot{v}_2}{\ddot{\mu}_2^2 + \ddot{v}_2^2}\right) + j\left(\frac{\ddot{v}_1\ddot{\mu}_2 - \ddot{\mu}_1\ddot{v}_2}{\ddot{\mu}_2^2 + \ddot{v}_2^2}\right)$$

The conjugate $\bar{\ddot{z}}$ is the reflection of $\ddot{z}$.

Definition 15: Let $\ddot{z} = \ddot{\mu} + j\ddot{v}$ be a complex function where the degree of membership is $\ddot{\mu}$ and the degree of non-membership is $\ddot{v}$. Then the reciprocal of $\ddot{z}$ is defined as:

$$\frac{1}{\ddot{z}} = \frac{\bar{\ddot{z}}}{\ddot{z}\bar{\ddot{z}}} = \frac{\ddot{\mu} - j\ddot{v}}{(\ddot{\mu} + j\ddot{v})(\ddot{\mu} - j\ddot{v})} = \frac{\ddot{\mu}}{\ddot{\mu}^2 + \ddot{v}^2} - j\frac{\ddot{v}}{\ddot{\mu}^2 + \ddot{v}^2}$$

Let $\ddot{z} = \ddot{\mu} + j\ddot{v}$ be a complex function where the degree of membership is $\ddot{\mu}$ and the degree of non-membership is $\ddot{v}$, then the following properties can be derived from definitions 12-15:

1. $\bar{\bar{\ddot{z}}} = \ddot{z}$,
2. $\overline{\ddot{z}_1 \pm \ddot{z}_2} = \bar{\ddot{z}}_1 \pm \bar{\ddot{z}}_2$,
3. $\overline{\ddot{z}_1\ddot{z}_2} = \bar{\ddot{z}}_1\bar{\ddot{z}}_2$,
4. $\overline{\left(\frac{\ddot{z}_1}{\ddot{z}_2}\right)} = \frac{\bar{\ddot{z}}_1}{\bar{\ddot{z}}_2}$,
5. $\ddot{z}\bar{\ddot{z}} = |z|^2$,
6. $|\ddot{z}| = |\bar{\ddot{z}}|$,
7. $|-\ddot{z}| = |\ddot{z}|$,
8. $\frac{\ddot{z}_1}{\ddot{z}_2} = \frac{\ddot{z}_1\bar{\ddot{z}}_2}{|\ddot{z}_2|^2}$,
9. $|\ddot{z}_1\ddot{z}_2| = |z_1||z_2|$,
10. $\left|\frac{\ddot{z}_1}{\ddot{z}_2}\right| = \frac{|\ddot{z}_1|}{|\ddot{z}_2|}$,
11. $|\ddot{z}_1 + \ddot{z}_2| \leq |\ddot{z}_1| + |\ddot{z}_2|$.

Definition 16: Let $\ddot{z} = \ddot{\mu} + j\ddot{v}$ be a complex function where the degree of membership is $\ddot{\mu}$ and the degree of non-membership is $\ddot{v}$. Then the absolute (modulus or magnitude) value of $\ddot{z}$ is $\ddot{r} = |\ddot{z}| = \sqrt{\ddot{\mu}^2 + \ddot{v}^2}$.

Theorem 3: Let Ä be an intuitionistic fuzzy set which is characterized by a complex function $\ddot{z} = \ddot{\mu} + j\ddot{v}$ where $\ddot{\mu}$ and $\ddot{v}$ are the degrees of membership and non-membership respectively. Then the square of the absolute (modulus or magnitude) value of $\ddot{z} = \ddot{\mu} + j\ddot{v}$ forms a Pythagorean fuzzy set $\ddot{A}_P$ [41].

Proof: Consider an intuitionistic fuzzy set Ä which is characterized by the complex function $\ddot{z} = \ddot{\mu} + j\ddot{v}$ where the degree of membership is $\ddot{\mu}$ and the degree of non-membership is $\ddot{v}$, where both degrees are non-negative and their sum is less than or equal to 1. Then the absolute value of $\ddot{z}$ is:

$$\ddot{r} = |\ddot{z}| = \sqrt{\ddot{\mu}^2 + \ddot{v}^2}.$$

Hence,

$$\ddot{r}^2 = |\ddot{z}|^2 = \ddot{\mu}^2 + \ddot{v}^2.$$

We know that $\ddot{\mu} + \ddot{v} \in [0,1]$. Hence,

$$0 \leq \ddot{\mu}^2 + \ddot{v}^2 \leq 1$$

which completes the proof. ■

Theorem 4: Let Ä be an intuitionistic fuzzy set which is characterized by a complex function $\ddot{z} = \ddot{\mu} + j\ddot{v}$ where $\ddot{\mu}$ and $\ddot{v}$ are the degrees of membership and non-membership respectively. Then the multiplication of $\ddot{z} = \ddot{\mu} + j\ddot{v}$ by its conjugate $\bar{\ddot{z}} = \ddot{\mu} - j\ddot{v}$ forms a Pythagorean fuzzy set $\ddot{A}_P$ [41].

Proof: The proof follows from Definition 12 and Theorem 3.

Definition 17: Let $\ddot{z} = \ddot{\mu} + j\ddot{v}$ be a complex function where the degree of membership is $\ddot{\mu}$ and the degree of non-membership is $\ddot{v}$. Then the argument is given by

$$\theta(x) = arg(\ddot{z})(x) = \begin{cases} \arctan\frac{\ddot{v}(x)}{\ddot{\mu}(x)} & when\ \ddot{\mu}(x) > 0, \\ \frac{\pi}{2} & when\ \ddot{\mu}(x) = 0\ and\ \ddot{v}(x) > 0. \end{cases}$$

The value of θ is the angle of $\ddot{r} = |\ddot{z}|$ along with the membership degree.

Theorem 6: The product of the absolute value $\ddot{r} = |\ddot{z}|$ and argument forms a complex fuzzy set $\ddot{A}_C$.

Proof: By multiplying the modulus value and argument, we have

$$\ddot{z} = \ddot{r}(cos\theta + jsin\theta),$$

By Euler's formula, we get

$$\ddot{z} = \ddot{r}e^{i\theta}.$$

By definition, setting $\ddot{r}$ as the amplitude term and θ as the phase term forms a complex fuzzy set. ■

V. Conclusion

In this paper, we utilized complex numbers to characterize an intuitionistic fuzzy set where the real part represents the membership function while the imaginary part represents the non-membership function. This characterization can be used as an extension to several concepts, such as fuzzy sets, Pythagorean fuzzy sets, and complex fuzzy sets. We discussed complex numbers-based set theoretic operations such as union, intersection, and complement. We defined the No-Man Zone (NMZ) and found the relation of the NMZ to complex number representation. Furthermore, we have introduced athematic operations of intuitionistic fuzzy sets via complex numbers. We have found that the square of the absolute of an intuitionistic fuzzy set becomes a Pythagorean fuzzy set. The polar form of intuitionistic fuzzy sets reduces to complex fuzzy sets. We plan to extend this in several ways. First, we plan to expand this work to intuitionistic fuzzy logic; additionally we plan to expand the use of complex number representation to represent complex intuitionistic fuzzy set theory and logic.

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