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Representing complex intuitionistic fuzzy set by quaternion numbers and applications to decision making

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ABSTRACT

Intuitionistic fuzzy sets are useful for modeling uncertain data of realistic problems. In this paper, we generalize and expand the utility of complex intuitionistic fuzzy sets using the space of quaternion numbers. The proposed representation can capture composite features and convey multi-dimensional fuzzy information via the functions of real membership, imaginary membership, real non-membership, and imaginary non-membership. We analyze the order relations and logic operations of the complex intuitionistic fuzzy set theory and introduce new operations based on quaternion numbers. We also present two quaternion distance measures in algebraic and polar forms and analyze their properties. We apply the quaternion representations and measures to decision-making models. The proposed model is experimentally validated in medical diagnosis, which is an emerging application for tackling patient's symptoms and attributes of diseases.

1. Introduction

In 1965, fuzzy sets (FSs) were introduced as an extension of the crisp sets by Zadeh [1], in order to address uncertainty and ambiguity. A membership function whose range is the [0, 1] interval defines a fuzzy set. Fuzzy logic and fuzzy sets have numerous applications in signal processing [2], control theory [3], and data mining [4–7]. Applications are of recent interest include the fuzzy linguistic modeling used in information accessing systems in order to increase flexibility [8] and the fuzzy S-tree applied to medical image retrieval [9]. Besides, some other nonlinear modeling approaches were proven effective in various problems, such as modeling derived from Bayesian filtering [10] or training Echo State Neural Network based on harmony search methods [11]. Further, they have significant applications in consistency and consensus processes [12–15] and also large scale decision problems and social networks [16,17]. In the present paper, we expand the fuzzy set theory into multi-dimensional space and consider an application in a medical diagnosis problem.

In 1986, intuitionistic fuzzy sets (IFSs) were proposed by Atanassov [18], extending the notion of an FS by adding a non-membership function. Overcoming the restrictions of fuzzy sets in handling conflicting information concerning membership of objects, this concept is used in modeling imprecision [19], pattern recognition [20], computational intelligence [21], and decision making [22]. Specifically, intuitionistic fuzzy sets are quite useful in medical applications, such as medical diagnosis problems [23] and medical image segmentation (breast cancer, dental X-ray images, etc.) [24,25].

Ramot et al. [26] introduced the CFS — complex fuzzy sets in 2002. The proposed formalism was based on the polar representation of complex numbers, where the amplitude is a fuzzy function and the phase is a general function [26]. CFSs are useful in solving complicated problems, such as multiple periodic factor prediction problems [27]. Alkouri et al. [28,29] used complex grades of membership and non-membership to construct a generalization of IFSs and CFSs called complex intuitionistic fuzzy set (CIFS). The initial approach for CIFS used the Ramot CFS [26], where the functions of membership and non-membership employ the Ramot-based complex fuzzy sets. The range of the complex degree of membership is a unit disk in a complex plane. A decision-making model using the distance measure of CIFSs was presented by Alkouri and Salleh [30], as an example of the theory.

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Nevertheless, Ramot's formalism and the derived CIFS [26,28] are limited to the polar representation, where the amplitude term is in the interval [0, 1], that conveys indistinct information [31]. To overcome this problem, Tamir et al. [31] proposed a new concept of complex fuzzy class (CFC) via pure complex fuzzy membership degree; here the range of both the real and imaginary components is the unit interval. Furthermore, Tamir et al. [31,32] successfully applied complex fuzzy classes to problems emerging from physics and stock markets. In 2016, Mumtaz et al. [33] applied the definitions proposed by Tamir et al. and extended the work presented in [31,32] to the new concept of complex intuitionistic fuzzy classes. One drawback of this formalism, however, is that the operations defined are complicated and not readily accessible.

Recently, Tamir et al. [34] introduced a method for complex number representation of IFS with important properties. Evolving from [34], in this paper, a generalization of IFSs, as presented in [34] is introduced by representing complex intuitionistic fuzzy sets in a quaternion number system, a number system that extends the complex number system. A quaternion number has the form of $a + bi + cj + dk$, where quaternion units i, j, k are the square roots of -1, complying with the conditions $i^2 = j^2 = k^2 = ijk = -1$, and a, b, c, d are real values. Although the quaternion multiplication has associative property, it has not commutative property. For examples, $jk = -kj = i$ and $i^2 = (jk)^2 = (jk) . (jk) = (jk) . (-kj) = -j\left(k^2\right) j = -j(-1) j = j^2 = -1$, hence $i^2 \neq j^2 k^2$. Quaternion number representation can capture composite features and convey fuzzy information in four dimensions rather than in two, as in the complex number representation.

Particularly, we use the concept of quaternion numbers to combine the degrees of complex membership with complex non-membership, i.e., combine the degrees of real membership, imaginary membership, real non-membership, with imaginary non-membership. The addition of these degrees of freedom provides a deep layer of expressiveness and enables efficient approach toward uncertainty and ambiguity. Moreover, this methodology enables flexibility in evaluating uncertain information from different aspects because quaternion numbers can be represented in two forms: algebraic form and polar form.

In this new development, we define order relations, set-theoretic operations, and some other operations of Quaternion Complex Intuitionistic Fuzzy Sets (CIFS-Q) and show their properties. We propose quaternion distance measures in both the algebraic form and polar form. Some of their important properties are shown. Moreover, in order to illustrate the applicability of the introduced approach, we build a new decision-making model for medical diagnosis based on the proposed representations and Quaternion Distance Measures (QDM). QDM is developed from the model proposed in our previous research [7]. The model addresses relevant problems via the quaternionification process, i.e., a pair of real and imaginary components is used to represent the input information, and makes diagnostic suggestions based on quaternion distance measures.

The main contributions of the QDM algorithm compared to the previous representation and distance measure methods (SM1-1, SM1-2, SM1-3, SM1-4 [35], SM2 [23], WXM [36], VSM [20], ZJM [37], WM [38,39], JM [40], SAM [41], and H-max [7]) include:

- QDM represents complex intuitionistic fuzzy information by quaternion numbers. This broadens the representability of fuzzy information.
- QDM introduces flexibility in evaluating information because the information can be represented in the algebraic (Cartesian) form (C-QDM) or in the polar form (P-QDM).
- In the IFSs generalization, the representation used in QDM via quaternion numbers is simpler than that of CFCs.
- In the QDM model, the quaternionification process is used for encoding information instead of the fuzzification process used in previous methods; thereby, it provides two more parameters for complex intuitionistic fuzzy sets representation for the handling of multidimensional fuzzy information.
- The QDM applies the Pearson correlation coefficient function for constructing the knowledge base in the polar form of quaternion numbers in the training process.
- New four-dimensional distance measures proposed for the QDM decision-making model, namely the Euclidean quaternion distance measure and $\theta-$distance measure, are empirically shown to be better than the existing fuzzy distance measures.

We apply the proposed model to medical diagnosis problem, with ability to handle highly complex patient's symptoms and attributes of diseases. Specifically, we have experimentally validated the QDM model on benchmark medical diagnosis datasets and have shown advantage in comparison to previously known methods.

The remainder of the paper includes the following sections. Background concepts are defined in Section 2. Section 3 introduces a new representation of CIFS based on quaternion numbers (CIFS-Q). Algebraic operations on CIFS-Qs are shown in Section 4. Section 5 studies quaternion distance measures. Section 6 introduces a new decision-making model based on the proposed quaternion distance measures. Experiments on benchmark medical diagnosis datasets are presented in Section 7. Finally, Section 8 draws conclusions and proposes further research.

2. Preliminary

In this section, we present basic definitions and related notions used in the paper. Under the early work on fuzzy set theory, a FS over a space $\ddot{X}$ has been defined via a function of membership on $\ddot{X}$ as follows:

Definition 1 (*[1]*)**.** A fuzzy set F over $\ddot{X}$ is formed by:

$$F = \left\{(x, \mu_F(x)) : x \in \ddot{X}\right\},$$

where $\mu_F(x) \in [0, 1]$ is the membership degree of x in F.

The IFSs theory introduced by Atanassov [18] adds a fuzzy function of non-membership to the fuzzy set in the following way:

Definition 2 (*[18,42]*)**.** An intuitionistic fuzzy set I over $\ddot{X}$ is formed by:

$$I = \left\{(x, \mu_I(x), \nu_I(x)) : x \in \ddot{X}\right\}, \tag{1}$$

where $\mu_I : \ddot{X} \to [0, 1]$ and $\nu_I : \ddot{X} \to [0, 1]$, the membership and non-membership functions, satisfy

$$0 \leq \mu_I + \nu_I \leq 1. \tag{2}$$

In order to clarify the proposals of this paper, the previously introduced definitions of representations of IFs and IFSs via complex numbers are mentioned as follows.

In 2002, Ramot et al. [26] introduced the concept of complex fuzzy set, where the membership function is the complex function in the polar form including the fuzzy amplitude function and the general phase function.

Definition 3 (*[26,43]*)**.** A complex fuzzy set S over X, is formed by

$$S = \{(x, \eta_S(x)) : x \in X\}, \tag{3}$$

where the complex-valued membership function, η_S, has the form $p_S.e^{j.\mu_S}$, here, $j = \sqrt{-1}$, the amplitude function, p_S, satisfies $p_S : X \to [0, 1]$, and the function μ_S is real-valued on X.

In 2012, Alkouri et al. [28,29] proposed the complex intuitionistic fuzzy set based on the Ramot CFS [26] by accreting the complex-valued non-membership function.

Furthermore, Tamir et al. [31] (2011) and Mumtaz et al. [33] (2016) proposed the complex fuzzy and complex intuitionistic fuzzy classes by combining the relation of an element and a set and the relation of a set and a class.

Definition 4 (*[31]*). A complex fuzzy class Γ over $\ddot{X}$ has the representation as follows

$$\Gamma = \left\{ (E, x, \mu_{\Gamma} (E, x)) : E \in 2^{\ddot{X}}, x \in \ddot{X} \right\}, \tag{4}$$

where $2^{\ddot{X}}$ is the power-set of $\ddot{X}$. The pure complex membership degree is the degree of membership of E in Γ and the membership degree of x in E, which is

$$\mu_{\Gamma} (E, x) = \mu_r (E) + j\mu_i (x), \tag{5}$$

where $\mu_r (E)$,$\mu_i (x)$$\in [0, 1]$.

Similarly, the concept of complex intuitionistic fuzzy class was introduced by adding to the concept of complex fuzzy class the pure complex non-membership degree [33].

In 2016, complex numbers in the Cartesian form were used more simply in the representations of IFSs.

Definition 5 (*Tamir et al. [34]*)**.** Let $\ddot{A}$ be an IFS characterized by the complex number function $\ddot{z} = \ddot{\mu} + j\ddot{\nu}$, where $\ddot{\mu}$, $\ddot{\nu}$: $\ddot{X} \to [0, 1]$ satisfying $\ddot{\mu} + \ddot{\nu} \in [0, 1]$ are the functions of membership and non-membership, respectively. As a set of ordered pairs, the IFS $\ddot{A}$ can be represented as:

$$\ddot{A} = \left\{ (\ddot{x}, \ddot{z}) | \, \ddot{x} \in \ddot{X}, \ddot{z} = \ddot{\mu} (x) + j\ddot{\nu} (x) \right\}. \tag{6}$$

Further, an IFS $\ddot{A}$ is defined to be a subset of an IFS $\ddot{B}$ iff $\ddot{z}_A \leq \ddot{z}_B$,i.e.

$$\ddot{A} \subseteq \ddot{B} \Leftrightarrow \ddot{\mu}_A (x) \leq \ddot{\mu}_B (x), \;\; \ddot{\nu}_A (x) \geq \ddot{\nu}_B (x), \forall x \in \ddot{X}. \tag{7}$$

3. New representation of complex intuitionistic fuzzy sets based on quaternion numbers

In 4-dimensional space, the quaternion number system was introduced by Hamilton in 1943 [44] as an extension of the complex number system. We propose to expand representation of complex intuitionistic fuzzy sets via quaternion numbers. This model advances the prior model [34] by using the quaternion numbers instead of complex numbers in representing intuitionistic fuzzy sets. Section 3 shows that the proposed model with four representative parameters is more powerful than the representation of IFSs based on complex numbers introduced in [34] with two representative parameters.

Definition 6. Let $\ddot{U}$ be a space. $\ddot{F}_Q$ is the complex intuitionistic fuzzy set on $\ddot{U}$ defined by a quaternion function (CIFS-Q) with the quaternion function $Q = \ddot{q} = \ddot{\alpha} + i\ddot{\beta} + j\ddot{\omega} + k\ddot{\gamma}$, where i, j, k are complex roots, $i^2 = j^2 = k^2 = ijk = -1$. Here, $\ddot{\alpha}$, $\ddot{\beta}$, $\ddot{\omega}$, and $\ddot{\gamma}$ are the functions of real membership, imaginary membership, real non-membership, and imaginary non-membership, respectively. For all $u \in \ddot{U}$, the functions $\ddot{\alpha}$, $\ddot{\beta}$, $\ddot{\omega}$ and $\ddot{\gamma}$ satisfy the following conditions:

$$\ddot{\alpha}(u), \ddot{\beta}(u), \ddot{\omega}(u), \ddot{\gamma}(u) \in [0, 1], \tag{8}$$

$$\ddot{\alpha}(u) + \ddot{\beta}(u) \leq 1, \tag{9}$$

$$\ddot{\omega}(u) + \ddot{\gamma}(u) \leq 1, \tag{10}$$

$$\ddot{\alpha}(u) + \ddot{\omega}(u) \leq 1, \tag{11}$$

$$\ddot{\beta}(u) + \ddot{\gamma}(u) \leq 1. \tag{12}$$

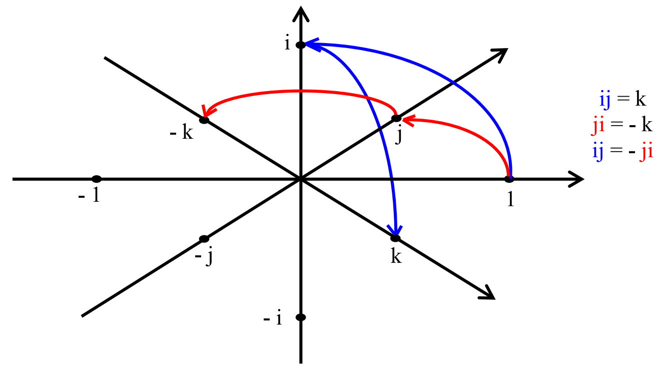

Fig. 1. Graphical representation of the products of unit quaternions as a 90°-rotation in 4D-space.

The values $\ddot{\alpha}(u)$, $\ddot{\beta}(u)$, $\ddot{\omega}(u)$, and $\ddot{\gamma}(u)$ are the degrees of real membership, imaginary membership, real non-membership, and imaginary non-membership, respectively, of u in $\ddot{F}_Q$. Then, the complex intuitionistic fuzzy set $\ddot{F}_Q$ represented by the quaternion function $\ddot{q}$ is

$$\ddot{F}_Q = \left\{ (u, Q(u)) : u \in \ddot{U} \right\}. \tag{13}$$

$Q = \ddot{q}$ is called the characteristic quaternion function of CIFS-Q. The quaternion function $Q = \ddot{q}$ is also written as follows:

$$\ddot{q} = \left(\ddot{\alpha} + i\ddot{\beta} \right) + j \left(\ddot{\omega} - i\ddot{\gamma} \right) = \ddot{\mu} + j\ddot{\nu}, \tag{14}$$

here, $\ddot{\mu} = \ddot{\alpha} + i\ddot{\beta}$ is the complex membership function, and $\ddot{\nu} = \ddot{\omega} - i\ddot{\gamma}$ is the complex non-membership function. For each $u \in \ddot{U}$, the values $\ddot{\mu}(u)$ and $\ddot{\nu}(u)$ are the complex membership and complex non-membership degrees of u in $\ddot{F}_Q$, respectively.

Definition 6 characterizes a CIFS on $\ddot{U}$ by a quaternion function Q. This is a direct extension of Definition 5. It can be seen that in (14) if $\ddot{\beta} = \ddot{\gamma} = 0$, then $\ddot{q} = \ddot{\mu} + j\ddot{\nu} = \ddot{\alpha} + j\ddot{\omega}$ where $\ddot{\alpha}, \ddot{\omega} \in [0, 1]$ and $\ddot{\alpha} + \ddot{\omega} \in [0, 1]$. Hence, the representation of IFSs based on complex numbers [34] in Definition 5 is a special case of the proposed model.

The following example serves as a way to check the inequality "$\ddot{\mu} + \ddot{\nu} \in [0, 1]$" of Definition 5 for the introduced sets, i.e., the inequalities (9)–(12) of Definition 6.

Example 1. Let $\ddot{F}_Q$ be a CIFS-Q on $\ddot{U}$ with $Q = \ddot{q} = 0.5 + 0.1i + 0.3j + 0.6k$. $\ddot{F}_Q$ indeed satisfies the conditions of a complex intuitionistic fuzzy set as follows $\ddot{\alpha} + \ddot{\beta} \leq 1$, $\ddot{\omega} + \ddot{\gamma} \leq 1$,$\ddot{\alpha} + \ddot{\omega} \leq 1$, and $\ddot{\beta} + \ddot{\gamma} \leq 1$,which are $0.5 + 0.1 = 0.6 \leq 1$, $0.3 + 0.6 = 0.9 \leq 1$,$0.5 + 0.3 = 0.8 \leq 1$, and $0.1 + 0.6 = 0.7 \leq 1$.

Remark 1. A quaternion function is defined to be $\ddot{q} = \ddot{\alpha} + i\ddot{\beta} + j\ddot{\omega} + k\ddot{\gamma}$, where i, j and k comply with the condition $i^2 = j^2 = k^2 = ijk = -1$. The products of unit quaternions are intuitively illustrated in Fig. 1.

The complex membership and complex non-membership functions in formula (14) are illustrated by Fig. 2.

The residual $2 - (\ddot{\alpha} + \ddot{\beta} + \ddot{\omega} + \ddot{\gamma})$ implies the existence of another fuzzy set referred as the No Man Zone (NMZ) of a CIFS-Q (see Definition 7).

Definition 7. Let $\ddot{F}_Q$ be a CIFS-Q with $Q = \ddot{q} = \ddot{\alpha} + i\ddot{\beta} + j\ddot{\omega} + k\ddot{\gamma}$, where $\ddot{\alpha}$, $\ddot{\beta}$, $\ddot{\omega}$, and $\ddot{\gamma}$ are the functions of real membership, imaginary membership, real non-membership, and imaginary non-membership, respectively. The NMZ of $\ddot{F}_Q$ is defined as:

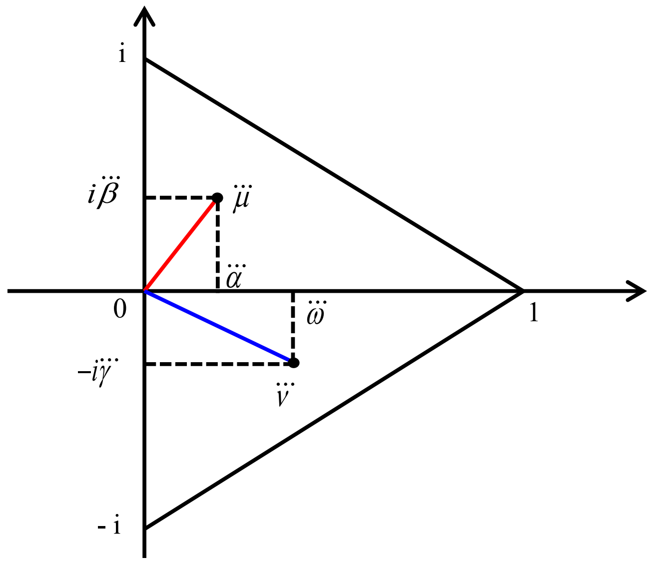

Fig. 2. Graphical representation of the complex degrees.

$$\mathrm{NMZ}(\dddot{F}_Q) = \left\{\left\langle u, 2-(\ddot{\alpha}+\ddot{\beta}+\ddot{\omega}+\ddot{\gamma})\right\rangle : u \in \ddot{U}\right\}. \tag{15}$$

Now, we consider the set Q^* defined by

$$Q^* = \{\ddot{q} = (\ddot{\alpha}, \ddot{\beta}, \ddot{\omega}, \ddot{\gamma}) = \ddot{\alpha} + i\ddot{\beta} + j\ddot{\omega} + k\ddot{\gamma} | \ddot{\alpha}, \ddot{\beta}, \ddot{\omega}, \ddot{\gamma}, \ddot{\alpha}+\ddot{\beta}, \ddot{\omega}+\ddot{\gamma}, \ddot{\alpha}+\ddot{\omega}, \ddot{\beta}+\ddot{\gamma} \in [0,1]\}.$$

Hereinafter it is assumed that if $\ddot{q}_t \in Q^*$ then $\ddot{q}_t$ has the representation $\ddot{q}_t = (\ddot{\alpha}_t, \ddot{\beta}_t, \ddot{\omega}_t, \ddot{\gamma}_t)$ or $\ddot{q}_t = \ddot{\alpha}_t + i\ddot{\beta}_t + j\ddot{\omega}_t + k\ddot{\gamma}_t, (t = 1, 2, \ldots)$.

The order relation on Q^* is defined by:

$$\ddot{q}_1 \le \ddot{q}_2 \Leftrightarrow \ddot{\alpha}_1 \le \ddot{\alpha}_2, \ddot{\beta}_1 \ge \ddot{\beta}_2, \ddot{\omega}_1 \ge \ddot{\omega}_2 \text{ and } \ddot{\gamma}_1 \le \ddot{\gamma}_2. \tag{16}$$

The units of Q^* are denoted as follows $1_{Q^*} = (1, 0, 0, 1)$ and $0_{Q^*} = (0, 1, 1, 0)$.

Definition 8. Let $Q_t = \ddot{q}_t = \ddot{\alpha}_t + i\ddot{\beta}_t + j\ddot{\omega}_t + k\ddot{\gamma}_t \in Q^*$ and $\dddot{F}_{Qt}$ be the CIFSs-Q on $\ddot{U}$, where $\ddot{\alpha}_t$,$\ddot{\beta}_t$,$\ddot{\omega}_t$,$\ddot{\gamma}_t$ $(t = 1; 2)$ are the functions of real membership, imaginary membership, real non-membership, and imaginary non-membership, respectively. The set $\dddot{F}_{Q1}$ is defined to be a subset of $\dddot{F}_{Q2}$, denoted as $\dddot{F}_{Q1} \subseteq \dddot{F}_{Q2}$, if and only if $\ddot{q}_1 \le \ddot{q}_2$.

Remark 2. In the proposed quaternion IFS model, the real membership and real non-membership degrees have the same meaning as the membership and non-membership degrees in Complex Number Representation of IFSs [34]. To illustrate this, let $A = \dddot{F}_Q$ be a CIFS-Q on $\ddot{U}$ with $Q = \ddot{q} = \ddot{\alpha} + i\ddot{\beta} + j\ddot{\omega} + k\ddot{\gamma}$, then A comprises a complex IFS characterized by the complex number function $\ddot{z} = \ddot{\alpha} + j\ddot{\omega}$. The proposed quaternion model extends the expressive power of IFSs to four degrees (the real membership, imaginary membership, real non-membership, and imaginary non-membership degrees), compared to the two degrees of complex IFSs (the membership and non-membership degrees according to [34]). If $A_{t(t=1,2)} = \dddot{F}_{Qt}$ are two CIFSs-Q on $\ddot{U}$, where each Q has the form $Q = \ddot{q} = \ddot{\alpha} + i\ddot{\beta} + j\ddot{\omega} + k\ddot{\gamma}$, and $\ddot{q}_1 \le \ddot{q}_2$, as defined by (16), then $\ddot{z}_1 \le \ddot{z}_2$, as defined by (7), where each $\ddot{z} = \ddot{\alpha} + j\ddot{\omega}$. Hence, the proposed quaternion representation allows to assess the relations between IFSs more closely with four dimensions. For example, let $u \in \ddot{U}$, $\ddot{q}_1(u) = 0.5 + 0.1i + 0.3j + 0.6k$, and $\ddot{q}_2(u) = 0.5 + 0.2i + 0.3j + 0.5k$. From (16) we have: $\ddot{q}_1(u) < \ddot{q}_2(u)$. However, from (7) we have: $\ddot{z}_1(u) = \ddot{z}_2(u) = 0.5 + 0.3j$.

The proposed quaternion representation of IFSs provides a broader multi-dimensional information representation capability. For example, voting results can be divided into three groups according to the number of voters that "vote for", "vote against", and "abstain from voting". This problem was approached by the traditional intuitionistic fuzzy theory via the membership, non-membership, and hesitancy degrees. There is also an identifiable category of voters who do not really want to support or oppose but they vote nevertheless due to secondary factors, such as following the crowd, voting unconsciously, etc. It can be observed that the previous representation of IFSs cannot model this aspect in a compact way, while the quaternion representation naturally models this phenomenon.

4. Logic and algebraic operations

In Section 4 logic operations and algebra operations based on the proposed representation are studied. Here, new logic operations are generalization of the previous intuitionistic fuzzy ones which were proposed by Atanassov [18].

Definition 9. A negation N on Q^* is a non-increasing $Q^* \to Q^*$ function that satisfies:

$$N(0_{Q^*}) = 1_{Q^*}, N(1_{Q^*}) = 0_{Q^*}. \tag{17}$$

A negation N is called an involution if $N(N(\ddot{q})) = \ddot{q}, \forall \ddot{q} \in Q^*$.

Proposition 1. *The following operation is an involutive negation on Q^*, called the standard negation on Q^*:*

$$N_s(\ddot{q}) = (\ddot{\omega}, \ddot{\gamma}, \ddot{\alpha}, \ddot{\beta}) = \ddot{\omega} + i\ddot{\gamma} + j\ddot{\alpha} + k\ddot{\beta}, \forall \ddot{q} \in Q^*. \tag{18}$$

Proof. Firstly, since $\ddot{q} \in Q^*$, therefore $\ddot{\omega} + \ddot{\gamma} \le 1$, $\ddot{\alpha} + \ddot{\beta} \le 1$,$\ddot{\omega} + \ddot{\alpha} \le 1$, and $\ddot{\gamma} + \ddot{\beta} \le 1$. Hence, we obtain $N(\ddot{q}) \in Q^*$. Secondly, we have $N_s(0_{Q^*}) = 1_{Q^*}$, $N_s(1_{Q^*}) = 0_{Q^*}$. Now, let $\ddot{q}_1 \le \ddot{q}_2$, that means $\ddot{\alpha}_1 \le \ddot{\alpha}_2$, $\ddot{\beta}_1 \ge \ddot{\beta}_2$, $\ddot{\omega}_1 \ge \ddot{\omega}_2$, and $\ddot{\gamma}_1 \le \ddot{\gamma}_2$, therefore $N(\ddot{q}_1) \ge N(\ddot{q}_2)$. Finally, we observe that $N_s(N_s(\ddot{q})) = \ddot{q}, \forall \ddot{q} \in Q^*$. □

Definition 10. Let $\dddot{F}_Q$ be a CIFS-Q on $\ddot{U}$ with $Q = \ddot{q} = \ddot{\alpha} + i\ddot{\beta} + j\ddot{\omega} + k\ddot{\gamma} \in Q^*$. The negation (the complement) of $\dddot{F}_Q$, denoted as $\dddot{F}_Q^N$, is the set represented by the complement of Q, i.e.,

$$Q^N = N(\ddot{q}), \tag{19}$$

where N is a negation on Q^*.

Remark 3. From Definition 10, we have $Q^{N_s} = N_s(\ddot{q}) = \ddot{\mu}^c + j\ddot{\nu}^c$, where $\ddot{\mu}^c = \ddot{\omega} + i\ddot{\gamma}$ and $\ddot{\nu}^c = \ddot{\alpha} - i\ddot{\beta}$ are the complex membership and complex non-membership functions of $\dddot{F}_Q^{N_s}$, respectively. From Proposition 1, we have $\left(\dddot{F}_Q^{N_s}\right)^{N_s} = \dddot{F}_Q$.

Definition 11. A t-norm T on Q^* is a $Q^* \times Q^* \to Q^*$ mapping and

1. $T(\ddot{q}, 1_{Q^*}) = \ddot{q}$ (the boundary condition),
2. $T(\ddot{q}_1, \ddot{q}_2) = T(\ddot{q}_2, \ddot{q}_1)$,
3. $T(\ddot{q}_1, T(\ddot{q}_2, \ddot{q}_3)) = T(T(\ddot{q}_1, \ddot{q}_2), \ddot{q}_3)$,
4. $T(\ddot{q}_1, \ddot{q}_2) \le T(\ddot{q}_1, \ddot{q}_3), \forall \ddot{q}_2 \le \ddot{q}_3$, here $\ddot{q}, \ddot{q}_1, \ddot{q}_2, \ddot{q}_3 \in Q^*$.

Definition 12. A t-conorm S on Q^* is a $Q^* \times Q^* \to Q^*$ mapping satisfying the boundary condition $S(\ddot{q}, 0_{Q^*}) = \ddot{q}, \forall \ddot{q} \in Q^*$ and the remaining conditions of a t-norm.

Proposition 2. *The following operations are t-norms on Q^*. For all $\dddot{q}_1, \dddot{q}_2 \in Q^*$,*

1. $T_1(\dddot{q}_1, \dddot{q}_2) = ((\dddot{\alpha}_1 \wedge \dddot{\alpha}_2), (\dddot{\beta}_1 \vee \dddot{\beta}_2), (\dddot{\omega}_1 \vee \dddot{\omega}_2), (\dddot{\gamma}_1 \wedge \dddot{\gamma}_2))$.
2. $T_2(\dddot{q}_1, \dddot{q}_2) = (\dddot{\alpha}_1\dddot{\alpha}_2, \dddot{\beta}_1+\dddot{\beta}_2-\dddot{\beta}_1\dddot{\beta}_2, \dddot{\omega}_1+\dddot{\omega}_2-\dddot{\omega}_1\dddot{\omega}_2, \dddot{\gamma}_1\dddot{\gamma}_2)$.
3. $T_3(\dddot{q}_1, \dddot{q}_2) = (0 \vee (\dddot{\alpha}_1 + \dddot{\alpha}_2 - 1), 1 \wedge (\dddot{\beta}_1 + \dddot{\beta}_2), 1 \wedge (\dddot{\omega}_1 + \dddot{\omega}_2), 0 \vee (\dddot{\gamma}_1 + \dddot{\gamma}_2 - 1))$.
4. $T_4(\dddot{q}_1, \dddot{q}_2) = (0 \vee (\dddot{\alpha}_1 + \dddot{\alpha}_2 - 1), \dddot{\beta}_1 + \dddot{\beta}_2 - \dddot{\beta}_1\dddot{\beta}_2, \dddot{\omega}_1 + \dddot{\omega}_2 - \dddot{\omega}_1\dddot{\omega}_2, 0 \vee (\dddot{\gamma}_1 + \dddot{\gamma}_2 - 1))$.

Proof. Firstly, we have to prove $T_1(\dddot{q}_1, \dddot{q}_2) \in Q^*$, i.e., $(\dddot{\alpha}_1 \wedge \dddot{\alpha}_2)+(\dddot{\beta}_1 \vee \dddot{\beta}_2) \leq 1$, $(\dddot{\omega}_1 \vee \dddot{\omega}_2)+(\dddot{\gamma}_1 \wedge \dddot{\gamma}_2) \leq 1$, $(\dddot{\alpha}_1 \wedge \dddot{\alpha}_2)+(\dddot{\omega}_1 \vee \dddot{\omega}_2) \leq 1$, and $(\dddot{\beta}_1 \vee \dddot{\beta}_2) + (\dddot{\gamma}_1 \wedge \dddot{\gamma}_2) \leq 1$. Secondly, we check the four conditions of Definition 11. Clearly, these properties are obtained from the validity of t-norm Min and t-conorm Max of the FSs defined by Zadeh [1] and the IFSs defined by Atanassov [5]. □

Proposition 3. *For all $\dddot{q}_1, \dddot{q}_2 \in Q^*$, the following operations are t-conorms on Q^*.*

1. $S_1(\dddot{q}_1, \dddot{q}_2) = ((\dddot{\alpha}_1 \vee \dddot{\alpha}_2), (\dddot{\beta}_1 \wedge \dddot{\beta}_2), (\dddot{\omega}_1 \wedge \dddot{\omega}_2), (\dddot{\gamma}_1 \vee \dddot{\gamma}_2))$.
2. $S_2(\dddot{q}_1, \dddot{q}_2) = (\dddot{\alpha}_1+\dddot{\alpha}_2-\dddot{\alpha}_1\dddot{\alpha}_2, \dddot{\beta}_1\dddot{\beta}_2, \dddot{\omega}_1\dddot{\omega}_2, \dddot{\gamma}_1+\dddot{\gamma}_2-\dddot{\gamma}_1\dddot{\gamma}_2)$.
3. $S_3(\dddot{q}_1, \dddot{q}_2) = (1 \wedge (\dddot{\alpha}_1 + \dddot{\alpha}_2), 0 \vee (\dddot{\beta}_1 + \dddot{\beta}_2 - 1), 0 \vee (\dddot{\omega}_1 + \dddot{\omega}_2 - 1), 1 \wedge (\dddot{\gamma}_1 + \dddot{\gamma}_2))$.
4. $S_4(\dddot{q}_1, \dddot{q}_2) = (\dddot{\alpha}_1 + \dddot{\alpha}_2 - \dddot{\alpha}_1\dddot{\alpha}_2, 0 \vee (\dddot{\beta}_1 + \dddot{\beta}_2 - 1), 0 \vee (\dddot{\omega}_1 + \dddot{\omega}_2 - 1), \dddot{\gamma}_1 + \dddot{\gamma}_2 - \dddot{\gamma}_1\dddot{\gamma}_2)$.

Proof. Similar to Proposition 2. □

Proposition 4. *Let t-norms t_1, t_2 and t-conorms s_1, s_2 satisfy the values of the sum$t_1 + s_1, t_2 + s_2, t_1 + s_2$, and $t_2 + s_1$ belong to* $[0, 1]$. *Then, the following operations are t-norm and t-conorm on Q^*:*

$$T(\dddot{q}_1, \dddot{q}_2) = (t_1(\dddot{\alpha}_1, \dddot{\alpha}_2), s_1(\dddot{\beta}_1, \dddot{\beta}_2), s_2(\dddot{\omega}_1, \dddot{\omega}_2), t_2(\dddot{\gamma}_1, \dddot{\gamma}_2)), \quad (20)$$

$$S(\dddot{q}_1, \dddot{q}_2) = (s_1(\dddot{\alpha}_1, \dddot{\alpha}_2), t_1(\dddot{\beta}_1, \dddot{\beta}_2), t_2(\dddot{\omega}_1, \dddot{\omega}_2), s_2(\dddot{\gamma}_1, \dddot{\gamma}_2)). \quad (21)$$

Definition 13. A t-norm T and a t-conorm S are called dual via a negation N if the triple (N, T, S) satisfies two following conditions

$$N(T(\dddot{q}_1, \dddot{q}_2)) = S(N(\dddot{q}_1), N(\dddot{q}_2)), \quad (22)$$

$$N(S(\dddot{q}_1, \dddot{q}_2)) = T(N(\dddot{q}_1), N(\dddot{q}_2)), \quad (23)$$

and then (N, T, S) is called a De Morgan triple on Q^*.

Proposition 5. *The (N_s, T_i, S_i) are De Morgan triples on Q^*, where $i = 1, 2, 3, 4$.*

Proof. We prove that (N_s, T_1, S_1) is a De Morgan triple on Q^*. Indeed, we have

$$N_s(T_1(\dddot{q}_1, \dddot{q}_2)) = ((\dddot{\omega}_1 \vee \dddot{\omega}_2), (\dddot{\gamma}_1 \wedge \dddot{\gamma}_2), (\dddot{\alpha}_1 \wedge \dddot{\alpha}_2), (\dddot{\beta}_1 \vee \dddot{\beta}_2)).$$

Since $N_s(\dddot{q}_t) = (\dddot{\omega}_t, \dddot{\gamma}_t, \dddot{\alpha}_t, \dddot{\beta}_t)(t = 1; 2)$, hence

$$S_1(N(\dddot{q}_1), N(\dddot{q}_2)) = ((\dddot{\omega}_1 \vee \dddot{\omega}_2), (\dddot{\gamma}_1 \wedge \dddot{\gamma}_2), (\dddot{\alpha}_1 \wedge \dddot{\alpha}_2), (\dddot{\beta}_1 \vee \dddot{\beta}_2)).$$

Therefore, $N_s(T_1(\dddot{q}_1, \dddot{q}_2)) = S_1(N_s(\dddot{q}_1), N_s(\dddot{q}_2))$.
Similarly, we obtain: $N_s(S_1(\dddot{q}_1, \dddot{q}_2)) = T_1(N_s(\dddot{q}_1), N_s(\dddot{q}_2))$.
Now, we prove that (N_s, T_2, S_2) is a De Morgan triple on Q^*. Indeed, we have

$$N_s(T_2(\dddot{q}_1, \dddot{q}_2)) = (\dddot{\omega}_1 + \dddot{\omega}_2 - \dddot{\omega}_1\dddot{\omega}_2, \dddot{\gamma}_1\dddot{\gamma}_2, \dddot{\alpha}_1\dddot{\alpha}_2, \dddot{\beta}_1 + \dddot{\beta}_2 - \dddot{\beta}_1\dddot{\beta}_2).$$

Since $N(\dddot{q}_t) = (\dddot{\omega}_t, \dddot{\gamma}_t, \dddot{\alpha}_t, \dddot{\beta}_t)(t = 1; 2)$,hence $N_s(T_2(\dddot{q}_1, \dddot{q}_2)) = S_2(N_s(\dddot{q}_1), N_s(\dddot{q}_2))$.
Similarly, we obtain (N_s, T_3, S_3) and (N_s, T_4, S_4) are De Morgan triples. □

Proposition 6. *Let t-norm t and t-conorm s satisfy the condition $t + s \in [0, 1]$, and*

$$T(\dddot{q}_1, \dddot{q}_2) = (t(\dddot{\alpha}_1, \dddot{\alpha}_2), s(\dddot{\beta}_1, \dddot{\beta}_2), s(\dddot{\omega}_1, \dddot{\omega}_2), t(\dddot{\gamma}_1, \dddot{\gamma}_2)), \quad (24)$$

$$S(\dddot{q}_1, \dddot{q}_2) = (s(\dddot{\alpha}_1, \dddot{\alpha}_2), t(\dddot{\beta}_1, \dddot{\beta}_2), t(\dddot{\omega}_1, \dddot{\omega}_2), s(\dddot{\gamma}_1, \dddot{\gamma}_2)), \quad (25)$$

then (N_s, T, S) is a De Morgan triple.

Proof. Similar to Proposition 5. □

Definition 14. Let $Q_t = \dddot{q}_t \in Q^*\,(t = 1; 2)$ be quaternion functions and $\dddot{F}_{Q1}, \dddot{F}_{Q2}$ be two CIFSs-Q on $\dddot{U}$. The intersection of $\dddot{F}_{Q1}$ and $\dddot{F}_{Q2}$ based on the t-norm T on Q^*, denoted as $\dddot{F}_{Q1} \cap_T \dddot{F}_{Q2}$, is the set represented by the function $T(\dddot{q}_1, \dddot{q}_2)$.

Definition 15. Let $Q_t = \dddot{q}_t \in Q^*\,(t = 1; 2)$ be quaternion functions and $\dddot{F}_{Q1}, \dddot{F}_{Q2}$ be two CIFSs-Q on $\dddot{U}$. The union of $\dddot{F}_{Q1}$ and $\dddot{F}_{Q2}$ based on the t-conorm S on Q^*, denoted as $\dddot{F}_{Q1} \cup_S \dddot{F}_{Q2}$, is the set represented by the function $S(\dddot{q}_1, \dddot{q}_2)$.

Example 2. Let $\dddot{F}_{Q1}$ and $\dddot{F}_{Q2}$ be two CIFSs-Q on $\dddot{U}$ with $Q_1 = \dddot{q}_1 = 0.5 + 0.1i + 0.3j + 0.6k$ and $Q_2 = \dddot{q}_2 = 0.2 + 0.1i + 0.1j + 0.4k$, respectively. Thus, $\dddot{F}_{Q1} \cap_{T_1} \dddot{F}_{Q2}$ and $\dddot{F}_{Q1} \cup_{S_1} \dddot{F}_{Q2}$ are complex intuitionistic fuzzy sets on $\dddot{U}$ defined by quaternion functions as follows:

$$\begin{aligned} T_1(\dddot{q}_1, \dddot{q}_2) &= (0.5 \wedge 0.2, 0.1 \vee 0.1, 0.3 \vee 0.1, 0.6 \wedge 0.4) \\ &= 0.2 + 0.1i + 0.3j + 0.4k, \\ S_1(\dddot{q}_1, \dddot{q}_2) &= (0.5 \vee 0.2, 0.1 \wedge 0.1, 0.3 \wedge 0.1, 0.6 \vee 0.4) \\ &= 0.5 + 0.1i + 0.1j + 0.6k. \end{aligned}$$

Remark 4. Let $Q_t = \dddot{q}_t \in Q^*\,(t = 1; 2)$ be quaternion functions and $\dddot{F}_{Q1}, \dddot{F}_{Q2}$ be two CIFSs-Q on $\dddot{U}$. For all t-norm T and t-conorm Son Q^*, clearly, $\dddot{F}_{Q1} \cap_T \dddot{F}_{Q2}$ and $\dddot{F}_{Q1} \cup_S \dddot{F}_{Q2}$ are two CIFSs-Q on $\dddot{U}$.

Proposition 7. *Let $Q_t = \dddot{q}_t \in Q^*\,(t = 1; 2; 3)$ be quaternion functions. Let $\dddot{F}_{Q1}, \dddot{F}_{Q2}$, and $\dddot{F}_{Q3}$ be three CIFSs-Q on $\dddot{U}$. It follows that, for all t-norm T, t-conorm S and negation N on Q^*,*

1. $\dddot{F}_{Q1} \cap_{T_1} \dddot{F}_{Q1} = \dddot{F}_{Q1}$, $\dddot{F}_{Q1} \cup_{S_1} \dddot{F}_{Q1} = \dddot{F}_{Q1}$ *(reflectivity property),*
2. $\dddot{F}_{Q1} \cap_T \dddot{F}_{Q2} = \dddot{F}_{Q2} \cap_T \dddot{F}_{Q1}$, $\dddot{F}_{Q1} \cup_S \dddot{F}_{Q2} = \dddot{F}_{Q2} \cup_S \dddot{F}_{Q1}$ *(commutative property),*
3. $(\dddot{F}_{Q1} \cap_T \dddot{F}_{Q2}) \cap_T \dddot{F}_{Q3} = \dddot{F}_{Q1} \cap_T (\dddot{F}_{Q2} \cap_T \dddot{F}_{Q3})$, $(\dddot{F}_{Q1} \cup_S \dddot{F}_{Q2}) \cup_S \dddot{F}_{Q3} = \dddot{F}_{Q1} \cup_S (\dddot{F}_{Q2} \cup_S \dddot{F}_{Q3})$ *(associative property),*
4. *If* $\dddot{F}_{Q1} \subseteq \dddot{F}_{Q2}$, *then* $\dddot{F}_{Q1} \cap_T \dddot{F}_{Q3} \subseteq \dddot{F}_{Q2} \cap_T \dddot{F}_{Q3}$ *and* $\dddot{F}_{Q1} \cup_S \dddot{F}_{Q3} \subseteq \dddot{F}_{Q2} \cup_S \dddot{F}_{Q3}$ *(monotone property),*
5. $(\dddot{F}_{Q1} \cup_{S_1} \dddot{F}_{Q2}) \cap_{T_1} \dddot{F}_{Q3} = (\dddot{F}_{Q1} \cap_{T_1} \dddot{F}_{Q3}) \cup_{S_1} (\dddot{F}_{Q2} \cap_{T_1} \dddot{F}_{Q3})$, $(\dddot{F}_{Q1} \cap_{T_1} \dddot{F}_{Q2}) \cup_{S_1} \dddot{F}_{Q3} = (\dddot{F}_{Q1} \cup_{S_1} \dddot{F}_{Q3}) \cap_{T_1} (\dddot{F}_{Q2} \cup_{S_1} \dddot{F}_{Q3})$ *(distributive property),*
6. *If (N, T, S) is a De Morgan triple, then* $(\dddot{F}_{Q1} \cap_T \dddot{F}_{Q2})^N = \dddot{F}^N_{Q1} \cup_S \dddot{F}^N_{Q2}$, $(\dddot{F}_{Q1} \cup_S \dddot{F}_{Q2})^N = \dddot{F}^N_{Q1} \cap_T \dddot{F}^N_{Q2}$ *(De Morgan's law).*

Proof. These properties are deduced from the properties of N, T, and S on Q^*. □

These set-theory operations are the generalization of previous set-theory operations on IFSs [18]. Now, let us consider other interesting operators along with their properties derived from properties of the quaternion numbers.

Definition 16. Let $\dddot{q}$ be a quaternion function defined by:

$$\dddot{q} = \dddot{\alpha} + i\dddot{\beta} + j\dddot{\omega} + k\dddot{\gamma} = (\dddot{\alpha} + i\dddot{\beta}) + j(\dddot{\omega} - i\dddot{\gamma}),$$

where the functions $\dddot{\mu} = \dddot{\alpha} + i\dddot{\beta}$ and $\dddot{\nu} = \dddot{\omega} - i\dddot{\gamma}$ are of complex membership and complex non-membership, respectively.

The norm of $\ddot{q}$ is,

$$\left|\ddot{q}\right| = \sqrt{\ddot{\alpha}^2 + \dddot{\beta}^2 + \ddot{\omega}^2 + \dddot{\gamma}^2}. \tag{26}$$

Definition 17. The conjugate of a quaternion function $\ddot{q}$ is,

$$\overline{\ddot{q}} = \ddot{\alpha} - i\dddot{\beta} - j\ddot{\omega} - k\dddot{\gamma} = (\ddot{\alpha} - i\dddot{\beta}) - j(\ddot{\omega} - i\dddot{\gamma}). \tag{27}$$

Definition 18. The inverse of a quaternion function $\ddot{q}$ is,

$$\ddot{q}^{-1} = \frac{\overline{\ddot{q}}}{\left|\ddot{q}\right|^2}. \tag{28}$$

The formula (28) makes intuitive sense when understanding some basic properties of the norm, as $\ddot{q} \times \overline{\ddot{q}} = \left|\ddot{q}\right|^2$.

Definition 19. Let $Q_t = \ddot{q}_t$ $(t = 1; 2)$ be quaternion functions and $\dddot{F}_{Q1}$, $\dddot{F}_{Q2}$ be two CIFSs-Q on $\ddot{U}$. The sum of $\dddot{F}_{Q1}$ and $\dddot{F}_{Q2}$ denoted as $\dddot{F}_{Q1} + \dddot{F}_{Q2}$, is

$$\dddot{F}_{Q1} + \dddot{F}_{Q2} = \left\{(u, (Q_1 + Q_2)(u)) : u \in \ddot{U}\right\}, \tag{29}$$

where

$$Q_1 + Q_2 = \ddot{q}_1 + \ddot{q}_2 = (\ddot{\alpha}_1 + \ddot{\alpha}_2) + i(\dddot{\beta}_1 + \dddot{\beta}_2) + j(\ddot{\omega}_1 + \ddot{\omega}_2) + k(\dddot{\gamma}_1 + \dddot{\gamma}_2). \tag{30}$$

Definition 20. Let $Q_t = \ddot{q}_t$ $(t = 1; 2)$ be quaternion functions and $\dddot{F}_{Q1}$, $\dddot{F}_{Q2}$ be two CIFSs-Q on $\ddot{U}$. The difference of $\dddot{F}_{Q1}$ less $\dddot{F}_{Q2}$ denoted $\dddot{F}_{Q1} - \dddot{F}_{Q2}$, is:

$$\dddot{F}_{Q1} - \dddot{F}_{Q2} = \left\{(u, (Q_1 - Q_2)(u)) : u \in \ddot{U}\right\}, \tag{31}$$

where

$$Q_1 - Q_2 = \ddot{q}_1 - \ddot{q}_2 = |\ddot{\alpha}_1 - \ddot{\alpha}_2| + i\left|\dddot{\beta}_1 - \dddot{\beta}_2\right| + j\left|\ddot{\omega}_1 - \ddot{\omega}_2\right| + k\left|\dddot{\gamma}_1 - \dddot{\gamma}_2\right|. \tag{32}$$

Definition 21. Let $Q_t = \ddot{q}_t$ $(t = 1; 2)$ be quaternion functions and $\dddot{F}_{Q1}$, $\dddot{F}_{Q2}$ be two CIFSs-Q on $\ddot{U}$. The product of $\dddot{F}_{Q1}$ and $\dddot{F}_{Q2}$ denoted $\dddot{F}_{Q1} \times \dddot{F}_{Q2}$, is:

$$\dddot{F}_{Q1} \times \dddot{F}_{Q2} = \left\{(u, (Q_1 \times Q_2)(u)) : u \in \ddot{U}\right\}, \tag{33}$$

where

$$\begin{aligned} Q_1 \times Q_2 = \ddot{q}_1 \times \ddot{q}_2 &= (\ddot{\alpha}_1\ddot{\alpha}_2 - \dddot{\beta}_1\dddot{\beta}_2 - \ddot{\omega}_1\ddot{\omega}_2 - \dddot{\gamma}_1\dddot{\gamma}_2) \\ &+ i(\ddot{\alpha}_1\dddot{\beta}_2 + \ddot{\alpha}_2\dddot{\beta}_1 + \ddot{\omega}_1\dddot{\gamma}_2 - \dddot{\gamma}_1\ddot{\omega}_2) \\ &+ j(\ddot{\alpha}_1\ddot{\omega}_2 - \dddot{\beta}_1\dddot{\gamma}_2 + \ddot{\omega}_1\ddot{\alpha}_2 + \dddot{\gamma}_1\dddot{\beta}_2) \\ &+ k(\ddot{\alpha}_1\dddot{\gamma}_2 + \dddot{\beta}_1\ddot{\omega}_2 - \ddot{\omega}_1\dddot{\beta}_2 + \dddot{\gamma}_1\ddot{\alpha}_2). \end{aligned} \tag{34}$$

Proposition 8. *Let $\ddot{q}_1$ and $\ddot{q}_2$ be the two characteristic quaternion functions of CIFSs-Q. The properties of the quaternion algebra are still preserved, such as $\overline{\overline{\ddot{q}}} = \ddot{q}$, $\overline{\ddot{q}_1 \pm \ddot{q}_2} = \overline{\ddot{q}_1} \pm \overline{\ddot{q}_2}$, $\overline{\ddot{q}_1 \times \ddot{q}_2} = \overline{\ddot{q}_1} \times \overline{\ddot{q}_2}$, and $\left(\ddot{q}_1 \times \ddot{q}_2\right)^{-1} = \ddot{q}_1^{-1} \times \ddot{q}_2^{-1}$.*

Proof. These assertions follow from Definitions 16–21

5. Quaternion distance measure

In Section 5 we propose and study the Euclidean quaternion distance measures in Cartesian form, a novel order relation, and a novel distance measure, which we have named θ-distance measure, in Polar form of the characteristic quaternion functions of CIFSs-Q.

5.1. Distance measure in Cartesian Form

Definition 22. Let $\ddot{q}_1, \ddot{q}_2 \in Q^*$ be two quaternion functions. The Euclidean quaternion distance measure between $\ddot{q}_1$ and $\ddot{q}_2$ is defined as:

$$\begin{aligned} d_E(\ddot{q}_1, \ddot{q}_2) &= \left|\ddot{q}_1 - \ddot{q}_2\right| \\ &= \sqrt{(\ddot{\alpha}_1 - \ddot{\alpha}_2)^2 + (\dddot{\beta}_1 - \dddot{\beta}_2)^2 + (\ddot{\omega}_1 - \ddot{\omega}_2)^2 + (\dddot{\gamma}_1 - \dddot{\gamma}_2)^2}. \end{aligned} \tag{35}$$

Proposition 9. *For all $\ddot{q}_1, \ddot{q}_2, \ddot{q}_3 \in Q^*$, the Euclidean quaternion distance measure satisfies the following features:*

1. $0 \le d_E(\ddot{q}_1, \ddot{q}_2)$, $d_E(\ddot{q}_1, \ddot{q}_2) = 0 \Leftrightarrow \ddot{q}_1 = \ddot{q}_2$,
2. $d_E(\ddot{q}_1, \ddot{q}_2) = d_E(\ddot{q}_2, \ddot{q}_1)$,
3. $d_E(\ddot{q}_1, \ddot{q}_2) + d_E(\ddot{q}_2, \ddot{q}_3) \ge d_E(\ddot{q}_1, \ddot{q}_3)$.
4. $\max\left(d_E(\ddot{q}_1, \ddot{q}_2), d_E(\ddot{q}_2, \ddot{q}_3)\right) \le d_E(\ddot{q}_1, \ddot{q}_3)$, *if* $\ddot{q}_1 \le \ddot{q}_2 \le \ddot{q}_3$, *the order in* Q^* *is defined by* (16).

Proof. Evidently,

$$\begin{aligned} d_E(\ddot{q}_1, \ddot{q}_3)^2 &= (\ddot{\alpha}_1 - \ddot{\alpha}_2)^2 + 2(\ddot{\alpha}_1 - \ddot{\alpha}_2)(\ddot{\alpha}_2 - \ddot{\alpha}_3) \\ &+ (\ddot{\alpha}_2 - \ddot{\alpha}_3)^2 + (\dddot{\beta}_1 - \dddot{\beta}_2)^2 \\ &+ 2(\dddot{\beta}_1 - \dddot{\beta}_2)(\dddot{\beta}_2 - \dddot{\beta}_3) + (\dddot{\beta}_2 - \dddot{\beta}_3)^2 \\ &+ (\ddot{\omega}_1 - \ddot{\omega}_2)^2 + 2(\ddot{\omega}_1 - \ddot{\omega}_2)(\ddot{\omega}_2 - \ddot{\omega}_3) \\ &+ (\ddot{\omega}_2 - \ddot{\omega}_3)^2 + (\dddot{\gamma}_1 - \dddot{\gamma}_2)^2 + 2(\dddot{\gamma}_1 - \dddot{\gamma}_2)(\dddot{\gamma}_2 - \dddot{\gamma}_3) \\ &+ (\dddot{\gamma}_2 - \dddot{\gamma}_3)^2. \end{aligned}$$

Furthermore, due to the Cauchy–Schwarz inequality, we have

$$\begin{aligned} &\Big((\ddot{\alpha}_1 - \ddot{\alpha}_2)(\ddot{\alpha}_2 - \ddot{\alpha}_3) + (\dddot{\beta}_1 - \dddot{\beta}_2)(\dddot{\beta}_2 - \dddot{\beta}_3) \\ &\quad + (\ddot{\omega}_1 - \ddot{\omega}_2)(\ddot{\omega}_2 - \ddot{\omega}_3) + (\dddot{\gamma}_1 - \dddot{\gamma}_2)(\dddot{\gamma}_2 - \dddot{\gamma}_3)\Big)^2 \\ &\le \left((\ddot{\alpha}_1 - \ddot{\alpha}_2)^2 + (\dddot{\beta}_1 - \dddot{\beta}_2)^2 + (\ddot{\omega}_1 - \ddot{\omega}_2)^2 + (\dddot{\gamma}_1 - \dddot{\gamma}_2)^2\right) \\ &\quad \times \Big((\ddot{\alpha}_2 - \ddot{\alpha}_3)^2 + (\dddot{\beta}_2 - \dddot{\beta}_3)^2 \\ &\quad + (\ddot{\omega}_2 - \ddot{\omega}_3)^2 + (\dddot{\gamma}_2 - \dddot{\gamma}_3)^2\Big). \end{aligned}$$

Hence, $d_E(\ddot{q}_1, \ddot{q}_3)^2 \le (d_E(\ddot{q}_1, \ddot{q}_2) + d_E(\ddot{q}_2, \ddot{q}_3))^2$.

Now, we obtain that $\ddot{\alpha}_1 \le \ddot{\alpha}_2 \le \ddot{\alpha}_3$, $\dddot{\beta}_1 \ge \dddot{\beta}_2 \ge \dddot{\beta}_3$, $\ddot{\omega}_1 \ge \ddot{\omega}_2 \ge \ddot{\omega}_3$, $\dddot{\gamma}_1 \le \dddot{\gamma}_2 \le \dddot{\gamma}_3$ from the order $\ddot{q}_1 \le \ddot{q}_2 \le \ddot{q}_3$. Therefore,

$$(\ddot{\alpha}_1 - \ddot{\alpha}_2)^2 \le (\ddot{\alpha}_1 - \ddot{\alpha}_3)^2, (\dddot{\beta}_1 - \dddot{\beta}_2)^2 \le (\dddot{\beta}_1 - \dddot{\beta}_3)^2,$$
$$(\ddot{\omega}_1 - \ddot{\omega}_2)^2 \le (\ddot{\omega}_1 - \ddot{\omega}_3)^2, \quad \text{and} \quad (\dddot{\gamma}_1 - \dddot{\gamma}_2)^2 \le (\dddot{\gamma}_1 - \dddot{\gamma}_3)^2.$$

Thus, we get the final property of Proposition 9. □

5.2. New order relation and distance measure based on polar form of quaternion numbers

Definition 23. Let $\ddot{q} = \ddot{\alpha} + i\dddot{\beta} + j\ddot{\omega} + k\dddot{\gamma} = \left(\ddot{\alpha} + i\dddot{\beta}\right) + j\left(\ddot{\omega} - i\dddot{\gamma}\right) = \ddot{\mu} + j\ddot{\nu} \in Q^*$ be a quaternion function on Q^*, then the Polar form of $\ddot{q}$ is:

$$\ddot{q} = \ddot{r}_\mu e^{i\theta_\mu} + j\ddot{r}_\nu e^{i\theta_\nu}, \tag{36}$$

where $\ddot{\mu} = \ddot{r}_\mu e^{i\theta_\mu}$, and $\ddot{\nu} = \ddot{r}_\nu e^{i\theta_\nu}$ are the polar forms of the complex functions $\ddot{\mu}$ and $\ddot{\nu}$, respectively. Here,

$$\ddot{r}_\mu = \left|\ddot{\mu}\right| = \sqrt{\ddot{\alpha}^2 + \dddot{\beta}^2}, \ddot{r}_\nu = |\ddot{\nu}| = \sqrt{\ddot{\omega}^2 + \dddot{\gamma}^2}. \tag{37}$$

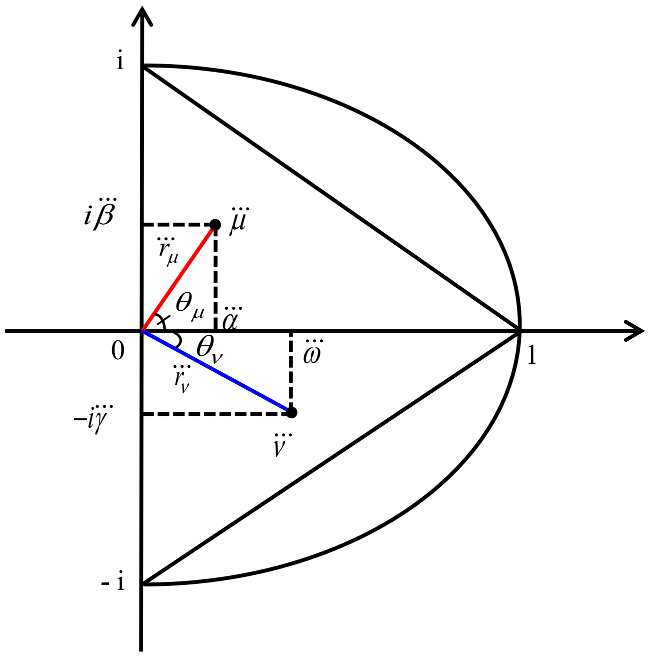

Fig. 3. The graphical representation of complex degrees in Polar form.

$$
\theta_\mu = \begin{cases} \arctan \dfrac{\dddot{\beta}}{\dddot{\alpha}} & \text{if } \dddot{\alpha} > 0 \\ \dfrac{\pi}{2} & \text{if } \dddot{\alpha} = 0, \dddot{\beta} > 0 \\ 0 & \text{if } \dddot{\alpha} = 0, \dddot{\beta} = 0 \end{cases},
$$
$$
\theta_\nu = \begin{cases} \arctan \dfrac{-\dddot{\gamma}}{\dddot{\omega}} & \text{if } \dddot{\omega} > 0 \\ -\dfrac{\pi}{2} & \text{if } \dddot{\omega} = 0, \dddot{\gamma} > 0 \\ 0 & \text{if } \dddot{\omega} = 0, \dddot{\gamma} = 0 \end{cases}. \tag{38}
$$

We can write the polar form of a quaternion function $\dddot{q}$ as:

$$
\dddot{q} = (\dddot{\mu}, \dddot{\nu}) = ([[\dddot{r}_\mu, \theta_\mu]], [[\dddot{r}_\nu, \theta_\nu]]), \tag{39}
$$

where $\dddot{r}_\mu$, $\dddot{r}_\nu$ are the modulus functions of $\dddot{\mu}$, $\dddot{\nu}$, and θ_μ, θ_ν are the argument functions of $\dddot{\mu}$, $\dddot{\nu}$, respectively, and $\dddot{\mu} = [[\dddot{r}_\mu, \theta_\mu]]$, $\dddot{\nu} = [[\dddot{r}_\nu, \theta_\nu]]$.

Note that $\dddot{r}_\mu = 0$ if and only if $\dddot{\alpha} = 0$ and $\dddot{\beta} = 0$, and in this case we conventionalize in the proposed representation of $\dddot{\mu}$ that $\theta_\mu = 0$. Hence, the pairs of the modulus $\dddot{r}_\mu = 0$ and the argument $\theta_\mu = 0$ have no meaning in this proposed representation. Likewise, we get the case of $\dddot{r}_\nu = 0$ in the proposed representation of $\dddot{\nu}$.

Fig. 3 represents the Polar form of the complex membership and complex non-membership degrees. Further, Figs. 2 and 3 in this paper and Figure 3.18 in the paper of Atanassov [45] in 2012 all mentioned the complex plane, but with different ideas. Herein, we discuss them as follows.

- Figs. 2 and 3 show a graphical representation of the complex membership degree $\dddot{\mu} = \dddot{\alpha} + i\dddot{\beta} = \dddot{r}_\mu e^{i\theta_\mu}$ and complex non-membership degree $\dddot{\nu} = \dddot{\omega} - i\dddot{\gamma} = j\dddot{r}_\nu e^{i\theta_\nu}$ of the representation of CIFS-Q. The relation negation of CIFS-Q in this paper (see Definition 9) cannot be completely analyzed in Figs. 2 and 3 because this relation is considered for points in 4D-space.
- Furthermore, Figure 3.18 in the paper of Atanassov in 2012 [45] is a geometrical representation of two conjugate points $a + ib$ and $a - ib$ in triangle ABC on the complex plane, where $A = (0, 1)$, $B = (1, 0)$, and $C = (0, -1)$. In that figure, triangle ABO is the geometrical interpretation of the intuitionistic fuzzy sets. Two elements $x = (x_1, x_2)$ and $y = (y_1, y_2)$ of triangle ABO are said to be in relation intuitionistic fuzzy negation if $x_1 = y_2$ and $x_2 = y_1$. Atanassov introduced the continuous bijective function $f(u, v)$ that transform the points of triangle ABC into triangle ABO, where (u, v) are the coordinates of the complex number $u + iv$ and $u \in [0, 1]$, $v \in [-1, 1]$. The interesting point is that for all the arbitrary complex conjugate numbers $a + ib$ and $a - ib$ (here$0 \le a, b, a + b \le 1$), $f(a, b)$ and $f(a, -b)$ are in the triangle ABO and share the intuitionistic fuzzy negation. Hence, Atanassov showed an open problem about interpretations of intuitionistic fuzzy negations through this analysis.

Note that $\dddot{q} = ([[\dddot{r}_\mu, \theta_\mu]], [[\dddot{r}_\nu, \theta_\nu]]) \in Q^*$ if and only if $\dddot{r}_\mu, \dddot{r}_\nu \in [0, 1]$, $\theta_\mu \in \left[0, \frac{\pi}{2}\right]$, $\theta_\nu \in \left[-\frac{\pi}{2}, 0\right]$, and

$$
\dddot{r}_\mu(\cos\theta_\mu + \sin\theta_\mu), \quad \dddot{r}_\nu(\cos\theta_\nu - \sin\theta_\nu) \in [0, 1], \tag{40}
$$

$$
\dddot{r}_\mu \cos\theta_\mu - \dddot{r}_\nu \sin\theta_\nu, \quad \dddot{r}_\nu \cos\theta_\nu + \dddot{r}_\mu \sin\theta_\mu \in [0, 1]. \tag{41}
$$

Now, a formula of the argument and modulus functions of CIFS-Q is introduced in the following example.

Example 3. Let the functions $\dddot{r}_\mu, \theta_\mu, \dddot{r}_\nu, \theta_\nu$ be defined as follows:

$$
\dddot{r}_\mu(x) = \begin{cases} 1 & \text{if } x < a \\ \left(\frac{b-x}{b-a}\right)^2 & \text{if } a \le x \le b \\ 0 & \text{if } x > b \end{cases},
$$
$$
\theta_\mu(x) = \begin{cases} \frac{\pi}{2} & \text{if } x < a \\ \frac{\pi}{2} \cdot \frac{b-x}{b-a} & \text{if } a \le x \le b \\ 0 & \text{if } x > b \end{cases},
$$
$$
\dddot{r}_\nu(x) = \begin{cases} 0 & \text{if } x < a \\ \left(\frac{x-a}{b-a}\right)^2 & \text{if } a \le x \le b \\ 1 & \text{if } x > b \end{cases},
$$
$$
\theta_\nu(x) = \begin{cases} 0 & \text{if } x < a \\ \frac{\pi}{2} \cdot \frac{a-x}{b-a} & \text{if } a \le x \le b \\ -\frac{\pi}{2} & \text{if } x > b \end{cases}
$$

Then the quaternion function $\dddot{q} = ([[\dddot{r}_\mu, \theta_\mu]], [[\dddot{r}_\nu, \theta_\nu]]) \in Q^*$.

Proof. Clearly, $\dddot{r}_\mu, \dddot{r}_\nu \in [0, 1]$, $\theta_\mu \in \left[0, \frac{\pi}{2}\right]$, $\theta_\nu \in \left[-\frac{\pi}{2}, 0\right]$. We have to prove that the functions $\dddot{r}_\mu, \theta_\mu, \dddot{r}_\nu, \theta_\nu$ satisfy the conditions (40) and (41). Indeed, these conditions are apparent in the cases $x < a$ and $x > b$. Now, let $a \le x \le b$, then

$$
x = \lambda a + (1 - \lambda) b, \lambda \in [0, 1].
$$

Hence, $\dddot{r}_\mu = \left(\frac{b-x}{b-a}\right)^2 = \lambda^2$, $\theta_\mu = \frac{\pi}{2}\lambda$, $\dddot{r}_\nu = (1-\lambda)^2$, and $-\theta_\nu = \frac{\pi}{2}(1-\lambda)$. We have

$$
\dddot{r}_\mu \cos\theta_\mu + \dddot{r}_\mu \sin\theta_\mu = \sqrt{2}\lambda^2 \cos\left(\frac{\pi}{2}\lambda - \frac{\pi}{4}\right).
$$

Consider the function $f(\lambda) = 1 - \sqrt{2}\lambda^2 \cos\left(\frac{\pi}{2}\lambda - \frac{\pi}{4}\right)$ on $[0, 1]$, we gain

$$
f'(\lambda) = 2\sqrt{2}\lambda \cos\left(\frac{\pi}{2}\lambda - \frac{\pi}{4}\right)\left[\frac{\pi}{4}\lambda \tan\left(\frac{\pi}{2}\lambda - \frac{\pi}{4}\right) - 1\right].
$$

Since, $\lambda \in [0,1]$, we have $\cos\left(\frac{\pi}{2}\lambda - \frac{\pi}{4}\right) > 0$, $\tan\left(\frac{\pi}{2}\lambda - \frac{\pi}{4}\right) \in [-1,1]$, and

$$\frac{\pi}{4}\lambda\tan\left(\frac{\pi}{2}\lambda - \frac{\pi}{4}\right) - 1 \in \left[-1, \frac{\pi}{4} - 1\right], \forall\lambda \in \left[\frac{1}{2}, 1\right],$$

$$\frac{\pi}{4}\lambda\tan\left(\frac{\pi}{2}\lambda - \frac{\pi}{4}\right) - 1 < 0, \forall\lambda \in \left[0, \frac{1}{2}\right].$$

Thus, $f'(\lambda) \le 0, \forall\lambda \in [0,1]$, hence f is a monotonically decreasing function on $[0,1]$ and $f(\lambda) \ge f(1) = 0$. Therefore, $0 \le \ddot{r}_\mu \cos\theta_\mu + \ddot{r}_\mu \sin\theta_\mu \le 1$.

Similarly, we obtain that $\ddot{r}_\nu \cos\theta_\nu - \ddot{r}_\nu \sin\theta_\nu = \sqrt{2}(1-\lambda)^2 \cos\left(\frac{\pi}{2}(1-\lambda) - \frac{\pi}{4}\right) \le 1$.

Now, since $\lambda \in [0,1]$ then $\lambda^2 \le \lambda$ and $(1-\lambda)^2 \le 1-\lambda$. Thus, $0 \le \ddot{r}_\mu + \ddot{r}_\nu \le 1$, and $\ddot{r}_\mu \cos\theta_\mu - \ddot{r}_\nu \sin\theta_\nu \le \ddot{r}_\mu + \ddot{r}_\nu \le 1$.

Thus, we obtain that $\ddot{q} = ([[\ddot{r}_\mu, \theta_\mu]], [[\ddot{r}_\nu, \theta_\nu]]) \in Q^*$. □

Now, based on the polar form of quaternion numbers, the other order relation on Q^*, denoted as $\le_*$, is proposed as follows.

Definition 24. Let $\ddot{q}_1 = ([[\ddot{r}_{\mu1}, \theta_{\mu1}]], [[\ddot{r}_{\nu1}, \theta_{\nu1}]])$, $\ddot{q}_2 = ([[\ddot{r}_{\mu2}, \theta_{\mu2}]], [[\ddot{r}_{\nu2}, \theta_{\nu2}]])$ be two quaternion functions, and $\ddot{q}_1, \ddot{q}_2 \in Q^*$. The order relation $\le_*$ on Q^* is defined by

$$\ddot{q}_1 \le_* \ddot{q}_2 \Leftrightarrow \begin{cases} [[\ddot{r}_{\mu1}, \theta_{\mu1}]] \le_* [[\ddot{r}_{\mu2}, \theta_{\mu2}]] \\ [[\ddot{r}_{\nu1}, \theta_{\nu1}]] \ge_* [[\ddot{r}_{\nu2}, \theta_{\nu2}]] \end{cases}$$
$$\Leftrightarrow \begin{cases} \left[\begin{array}{l} \theta_{\mu1} > \theta_{\mu2} \\ \theta_{\mu1} = \theta_{\mu2} \le \frac{\pi}{4}, \ddot{r}_{\mu1} \le \ddot{r}_{\mu2} \\ \theta_{\mu1} = \theta_{\mu2} > \frac{\pi}{4}, \ddot{r}_{\mu1} \ge \ddot{r}_{\mu2} \end{array}\right. \\ \left[\begin{array}{l} \theta_{\nu1} > \theta_{\nu2} \\ \theta_{\nu1} = \theta_{\nu2} \ge -\frac{\pi}{4}, \ddot{r}_{\nu1} \le \ddot{r}_{\nu2} \\ \theta_{\nu1} = \theta_{\nu2} < -\frac{\pi}{4}, \ddot{r}_{\nu1} \ge \ddot{r}_{\nu2} \end{array}\right. \end{cases}, \quad (42)$$

$$\ddot{q}_1 =_* \ddot{q}_2 \Leftrightarrow \begin{cases} [[\ddot{r}_{\mu1}, \theta_{\mu1}]] =_* [[\ddot{r}_{\mu2}, \theta_{\mu2}]] \\ [[\ddot{r}_{\nu1}, \theta_{\nu1}]] =_* [[\ddot{r}_{\nu2}, \theta_{\nu2}]] \end{cases}$$
$$\Leftrightarrow \begin{cases} \left[\begin{array}{l} \theta_{\mu1} = \theta_{\mu2} = \frac{\pi}{4} \\ \theta_{\mu1} = \theta_{\mu2}, \ddot{r}_{\mu1} = \ddot{r}_{\mu2} \end{array}\right. \\ \left[\begin{array}{l} \theta_{\nu1} = \theta_{\nu2} = -\frac{\pi}{4} \\ \theta_{\nu1} = \theta_{\nu2}, \ddot{r}_{\nu1} = \ddot{r}_{\nu2} \end{array}\right. \end{cases}. \quad (43)$$

Note that $1_{Q^*} = (1,0,0,1) = ([[1,0]], [[1, -\frac{\pi}{2}]])$ and $0_{Q^*} = (0,1,1,0) = ([[1, \frac{\pi}{2}]], [[1,0]])$ are still the units of Q^* within the order $\le_*$.

Remark 5. In Fig. 4, the red and blue arrows show the direction of increase of the elements on the segments which belong to two sides of the OM and ON.

Proposition 10. *Let $\ddot{q}_1, \ddot{q}_2 \in Q^*$ and $\ddot{q}_1 \le \ddot{q}_2$, defined by (16), then $\ddot{q}_1 \le_* \ddot{q}_2$, defined by (42).*

Proof. Let $\ddot{q}_1, \ddot{q}_2 \in Q^*$ and $\ddot{q}_1 \le \ddot{q}_2$, where

$$\ddot{q}_1 = \ddot{\alpha}_1 + i\dddot{\beta}_1 + j\dddot{\omega}_1 + k\dddot{\gamma}_1 = ([[\ddot{r}_{\mu1}, \theta_{\mu1}]], [[\ddot{r}_{\nu1}, \theta_{\nu1}]]),$$

$$\ddot{q}_2 = \ddot{\alpha}_2 + i\dddot{\beta}_2 + j\dddot{\omega}_2 + k\dddot{\gamma}_2 = ([[\ddot{r}_{\mu2}, \theta_{\mu2}]], [[\ddot{r}_{\nu2}, \theta_{\nu2}]]).$$

We obtain that $\ddot{\alpha}_1 \le \ddot{\alpha}_2, \dddot{\beta}_1 \ge \dddot{\beta}_2, \dddot{\omega}_1 \ge \dddot{\omega}_2$, and $\dddot{\gamma}_1 \le \dddot{\gamma}_2$. We consider three cases as follows:

Case 1: $0 < \ddot{\alpha}_1 \le \ddot{\alpha}_2$. We have $[[\ddot{r}_{\mu1}, \theta_{\mu1}]] \le_* [[\ddot{r}_{\mu2}, \theta_{\mu2}]]$. Indeed,

$$\theta_{\mu1} = \arctan\frac{\dddot{\beta}_1}{\ddot{\alpha}_1} \ge \theta_{\mu2} = \arctan\frac{\dddot{\beta}_2}{\ddot{\alpha}_2},$$

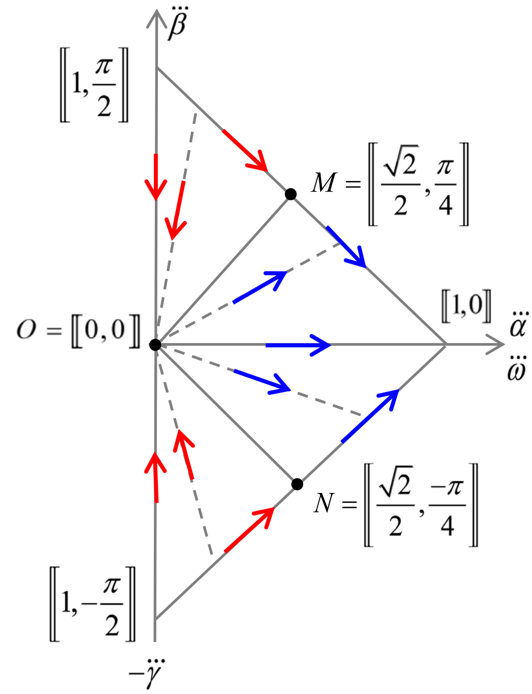

Fig. 4. The order relation $\le_*$ on Q^*.

$\theta_{\mu1} = \theta_{\mu2} \Leftrightarrow \ddot{\alpha}_1 = \ddot{\alpha}_2, \dddot{\beta}_1 = \dddot{\beta}_2 \Rightarrow \ddot{r}_{\mu1} = \ddot{r}_{\mu2}$.

Case 2: $0 = \ddot{\alpha}_1 < \ddot{\alpha}_2$. We also have $[[\ddot{r}_{\mu1}, \theta_{\mu1}]] \le_* [[\ddot{r}_{\mu2}, \theta_{\mu2}]]$. Indeed,

- If $\dddot{\beta}_1 > 0$, then $\theta_{\mu1} = \frac{\pi}{2} > \theta_{\mu2} = \arctan\frac{\dddot{\beta}_2}{\ddot{\alpha}_2}$.
- If $\dddot{\beta}_1 = 0$, then $\theta_{\mu1} = 0$. Since $\dddot{\beta}_1 \ge \dddot{\beta}_2$, hence $\dddot{\beta}_2 = 0, \theta_{\mu1} = \theta_{\mu2} = 0 < \frac{\pi}{4}$, and $\ddot{r}_{\mu1} < \ddot{r}_{\mu2}$.

Case 3: $0 = \ddot{\alpha}_1 = \ddot{\alpha}_2$. We have

- If $\dddot{\beta}_1 \ge \dddot{\beta}_2 > 0$, then $\theta_{\mu1} = \theta_{\mu2} = \frac{\pi}{2} > \frac{\pi}{4}$ and $\ddot{r}_{\mu1} \ge \ddot{r}_{\mu2}$.
- If $\dddot{\beta}_1 = \dddot{\beta}_2 = 0$, then $\theta_{\mu1} = \theta_{\mu2} = 0$ and $\ddot{r}_{\mu1} = \ddot{r}_{\mu2} = 0$.
- If $\dddot{\beta}_1 > \dddot{\beta}_2 = 0$, then $\theta_{\mu1} = \frac{\pi}{2} > \theta_{\mu2} = 0$.

Therefore, we always have $[[\ddot{r}_{\mu1}, \theta_{\mu1}]] \le_* [[\ddot{r}_{\mu2}, \theta_{\mu2}]]$. Similarly, we obtain that

$[[\ddot{r}_{\nu1}, \theta_{\nu1}]] \ge_* [[\ddot{r}_{\nu2}, \theta_{\nu2}]]$.

Thus, $\ddot{q}_1 \le_* \ddot{q}_2$ (see. (42)). □

Now, we introduce the new distance measure based on the order relation $\le_*$ on Q^*.

Definition 25. Let $\ddot{q}_1 = ([[\ddot{r}_{\mu1}, \theta_{\mu1}]], [[\ddot{r}_{\nu1}, \theta_{\nu1}]])$, $\ddot{q}_2 = ([[\ddot{r}_{\mu2}, \theta_{\mu2}]], [[\ddot{r}_{\nu2}, \theta_{\nu2}]])$ be two quaternion functions, and $\ddot{q}_1, \ddot{q}_2 \in Q^*$. The θ-distance measure on Q^* is:

$$d_\theta(\ddot{q}_1, \ddot{q}_2) = \frac{1}{2}(d_{1\theta}(\dddot{\mu}_1, \dddot{\mu}_2) + d_{2\theta}(\dddot{\nu}_1, \dddot{\nu}_2)), \quad (44)$$

where

$$d_{1\theta}(\dddot{\mu}_1, \dddot{\mu}_2) = \begin{cases} 1 - \cos\left(\frac{1}{2}\left|\theta_{\mu1} - \theta_{\mu2}\right| + \frac{\pi}{4}\right) \text{ if } \theta_{\mu1} \ne \theta_{\mu2} \\ \left|\ddot{r}_{\mu1} - \ddot{r}_{\mu2}\right|\left(1 - \cos\left|\theta_\mu - \frac{\pi}{4}\right|\right) \text{ if } \theta_{\mu1} = \theta_{\mu2} = \theta_\mu \end{cases}, \quad (45)$$

$$d_{2\theta}(\dddot{\nu}_1, \dddot{\nu}_2) = \begin{cases} 1-\cos\left(\frac{1}{2}\left|\theta_{\nu 1}-\theta_{\nu 2}\right|+\frac{\pi}{4}\right) \text{ if } \theta_{\nu 1} \neq \theta_{\nu 2} \\ \left|\dddot{r}_{\nu 1}-\dddot{r}_{\nu 2}\right|\left(1-\cos\left|\theta_{\nu}+\frac{\pi}{4}\right|\right) \text{ if } \theta_{\nu 1}=\theta_{\nu 2}=\theta_{\nu} \end{cases}. \quad (46)$$

Proposition 11. *The θ-distance measure satisfies the following properties. For all $\dddot{q}_{i(i=1,2,3)} = ([[\dddot{r}_{\mu i}, \theta_{\mu i}]], [[\dddot{r}_{\nu i}, \theta_{\nu i}]]) \in Q^*$,*

(1) $d_\theta(\dddot{q}_1, \dddot{q}_2) \in [0, 1]$.
(2) $d_\theta(\dddot{q}_1, \dddot{q}_2) = 0 \Leftrightarrow \dddot{q}_1 =_* \dddot{q}_2$, $d_\theta(1_{Q^*}, 0_{Q^*}) = 1$.
(3) $d_\theta(\dddot{q}_1, \dddot{q}_2) = d_\theta(\dddot{q}_2, \dddot{q}_1)$.
(4) $\max\left(d_\theta(\dddot{q}_1, \dddot{q}_2), d_\theta(\dddot{q}_2, \dddot{q}_3)\right) \leq d_\theta(\dddot{q}_1, \dddot{q}_3)$, *if* $\dddot{q}_1 \leq_* \dddot{q}_2 \leq_* \dddot{q}_3$,

Proof.

(1) Evidently, $d_{1\theta}(\dddot{\mu}_1, \dddot{\mu}_2), d_{2\theta}(\dddot{\nu}_1, \dddot{\nu}_2) \in [0, 1]$, hence $d_\theta(\dddot{q}_1, \dddot{q}_2) \in [0, 1]$.
(2) We have

$$d_{1\theta}(\dddot{\mu}_1, \dddot{\mu}_2) = d_{2\theta}(\dddot{\nu}_1, \dddot{\nu}_2) = 0 \Leftrightarrow \begin{cases} \left[\begin{array}{l} \theta_{\mu 1}=\theta_{\mu 2}=\frac{\pi}{4} \\ \theta_{\mu 1}=\theta_{\mu 2}, \dddot{r}_{\mu 1}=\dddot{r}_{\mu 2} \end{array}\right. \\ \left[\begin{array}{l} \theta_{\nu 1}=\theta_{\nu 2}=-\frac{\pi}{4} \\ \theta_{\nu 1}=\theta_{\nu 2}, \dddot{r}_{\nu 1}=\dddot{r}_{\nu 2} \end{array}\right. \end{cases}$$

$$\Leftrightarrow \dddot{q}_1 =_* \dddot{q}_2.$$

We also have

$$d_\theta(1_{Q^*}, 0_{Q^*}) = d_\theta(([[1,0]],[[1,-\frac{\pi}{2}]]),([[1,\frac{\pi}{2}]],[[1,0]]))$$
$$= \frac{1}{2}(d_{1\theta}([[1,0]],[[1,\frac{\pi}{2}]]) + d_{2\theta}([[1,-\frac{\pi}{2}]],[[1,0]]))$$
$$= 1-\cos(\frac{1}{2}.\frac{\pi}{2}+\frac{\pi}{4}) = 1.$$

(3) Evidently, d_θ has commutative property.
(4) Let $\dddot{q}_1 \leq_* \dddot{q}_2 \leq_* \dddot{q}_3$, then $\theta_{\mu 1} \geq \theta_{\mu 2} \geq \theta_{\mu 3}$ and $\theta_{\nu 1} \geq \theta_{\nu 2} \geq \theta_{\nu 3}$. Now, we prove that $\max\left(d_{1\theta}(\dddot{\mu}_1, \dddot{\mu}_2), d_{1\theta}(\dddot{\mu}_2, \dddot{\mu}_3)\right) \leq d_{1\theta}(\dddot{\mu}_1, \dddot{\mu}_3)$. Indeed, there are four cases as follows:

- Case 1: $\theta_{\mu 1} > \theta_{\mu 2} > \theta_{\mu 3}$. We have $\theta_{\mu 1}-\theta_{\mu 2} < \theta_{\mu 1}-\theta_{\mu 3}$

$$\Rightarrow \frac{\pi}{4} < \frac{1}{2}(\theta_{\mu 1}-\theta_{\mu 2})+\frac{\pi}{4} < \frac{1}{2}(\theta_{\mu 1}-\theta_{\mu 3})+\frac{\pi}{4} \leq \frac{\pi}{2}$$
$$\Rightarrow \cos(\frac{1}{2}(\theta_{\mu 1}-\theta_{\mu 2})+\frac{\pi}{4}) > \cos(\frac{1}{2}(\theta_{\mu 1}-\theta_{\mu 3})+\frac{\pi}{4})$$
$$\Rightarrow d_{1\theta}(\dddot{\mu}_1, \dddot{\mu}_2) = 1-\cos(\frac{1}{2}(\theta_{\mu 1}-\theta_{\mu 2})+\frac{\pi}{4})$$
$$< d_{1\theta}(\dddot{\mu}_1, \dddot{\mu}_3) = 1-\cos(\frac{1}{2}(\theta_{\mu 1}-\theta_{\mu 3})+\frac{\pi}{4}).$$

Similarly, $d_{1\theta}(\dddot{\mu}_2, \dddot{\mu}_3) < d_{1\theta}(\dddot{\mu}_1, \dddot{\mu}_3)$.

- Case 2: $\theta_{\mu 1} > \theta_{\mu 2} = \theta_{\mu 3}$. We have $\theta_{\mu 1}-\theta_{\mu 2} = \theta_{\mu 1}-\theta_{\mu 3}$, hence

$$d_{1\theta}(\dddot{\mu}_1, \dddot{\mu}_2) = d_{1\theta}(\dddot{\mu}_1, \dddot{\mu}_3) = 1-\cos(\frac{1}{2}(\theta_{\mu 1}-\theta_{\mu 3})+\frac{\pi}{4}),$$

and $d_{1\theta}(\dddot{\mu}_2, \dddot{\mu}_3) = \left|\dddot{r}_{\mu 2}-\dddot{r}_{\mu 3}\right|\left(1-\cos\left|\theta_{\mu 3}-\frac{\pi}{4}\right|\right) \leq 1-\cos\left|\theta_{\mu 3}-\frac{\pi}{4}\right|$. Since $\left|\theta_{\mu 3}-\frac{\pi}{4}\right| \in \left[0,\frac{\pi}{4}\right]$, $\frac{1}{2}(\theta_{\mu 1}-\theta_{\mu 3})+\frac{\pi}{4} \in \left(\frac{\pi}{4},\frac{\pi}{2}\right]$, hence

$$\cos\left|\theta_{\mu 3}-\frac{\pi}{4}\right| > \cos(\frac{1}{2}(\theta_{\mu 1}-\theta_{\mu 3})+\frac{\pi}{4})$$
$$\Rightarrow d_{1\theta}(\dddot{\mu}_2, \dddot{\mu}_3) < d_{1\theta}(\dddot{\mu}_1, \dddot{\mu}_3).$$

- Case 3: $\theta_{\mu 1} = \theta_{\mu 2} > \theta_{\mu 3}$. Similar to Case 2.

- Case 4: $\theta_{\mu 1} = \theta_{\mu 2} = \theta_{\mu 3}$. Evidently, $\max\left(d_{1\theta}(\dddot{\mu}_1, \dddot{\mu}_2), d_{1\theta}(\dddot{\mu}_2, \dddot{\mu}_3)\right) \leq d_{1\theta}(\dddot{\mu}_1, \dddot{\mu}_3)$.since

$$\left[\begin{array}{l} \dddot{r}_{\mu 1} \leq \dddot{r}_{\mu 2} \leq \dddot{r}_{\mu 3} \\ \dddot{r}_{\mu 1} \geq \dddot{r}_{\mu 2} \geq \dddot{r}_{\mu 3} \end{array}\right..$$

Similarly, we have $\max\left(d_{2\theta}(\dddot{\nu}_1, \dddot{\nu}_2), d_{2\theta}(\dddot{\nu}_2, \dddot{\nu}_3)\right) \leq d_{2\theta}(\dddot{\nu}_1, \dddot{\nu}_3)$. □

Table 1
m records of a dataset encoded in the Cartesian form of quaternion numbers.

	S_1	…	S_l	…	S_n	Class Y
p_1	$(\dddot{\alpha}_{11}, \dddot{\beta}_{11}, \dddot{\omega}_{11}, \dddot{\gamma}_{11})$	…	…	…	$(\dddot{\alpha}_{1n}, \dddot{\beta}_{1n}, \dddot{\omega}_{1n}, \dddot{\gamma}_{1n})$	y_1
…	…	…	…	…	…	…
p_m	$(\dddot{\alpha}_{m1}, \dddot{\beta}_{m1}, \dddot{\omega}_{m1}, \dddot{\gamma}_{m1})$	…	…	…	$(\dddot{\alpha}_{mn}, \dddot{\beta}_{mn}, \dddot{\omega}_{mn}, \dddot{\gamma}_{mn})$	y_m

6. Decision making model based on Quaternion Distance Measures

In this section, a new decision-making model using Quaternion Distance Measure, denoted by QDM, is shown. This model has evolved from the model used in [7], where the fuzzification functions and the H-max measure are replaced by the quaternionification functions and the quaternion distance measures, respectively, which are studied in Section 5. The logic operations proposed in Section 4 could be used to aggregate information when there are many different information flows.

A new decision-making model in medical diagnosis problem is illustrated by the diagram in Fig. 5. The QDM model contains three main parts:

i. Encoding the attributes of patients by CIFS-Q
ii. Formulating and encoding the attributes of the disease by CIFS-Q
iii. Calculating the relations between the patients and the disease based on the distance measure and suggesting likely diagnoses

The QDM model is performed in two different forms: C-QDM and P-QDM. They are described in detail as follows.

- **C-QDM Method:**

1. *Quaternionification*: Consider a medical dataset with m records of m corresponding patients p_i $(i = 1, \ldots, m)$ and with n attributes of a disease D.
Table 1 shows m records of the dataset encoded in the polar form of quaternion numbers. This step means determining and encoding relations between patients and attributes via quaternion numbers. Here, we obtain the relations between patient p_i and attributes S_l,$(\dddot{\alpha}_{i1}, \dddot{\beta}_{i1}, \dddot{\omega}_{i1}, \dddot{\gamma}_{i1})$. To elaborate, on each attribute $S_l(l = 1, \ldots, n)$, all the values of p_{il} (attributes characteristic for the patients p_i) are quaternionized (i.e., encoded) in the Cartesian form by the functions of real membership, imaginary membership, real non-membership, and imaginary non-membership, respectively, calculated as follows:

$$\dddot{\alpha}_l(x) = \begin{cases} 0 & \text{if } x < a_l \\ \frac{x-a_l}{b_l-a_l} & \text{if } a_l \leq x \leq b_l \\ 1 & \text{if } x > b_l \end{cases}, \quad (47)$$

$$\dddot{\beta}_l(x) = \begin{cases} 1 & \text{if } x < a_l' \\ \frac{b_l'-x}{b_l'-a_l'} & \text{if } a_l' \leq x \leq b_l' \\ 0 & \text{if } x > b_l' \end{cases}, \quad (48)$$

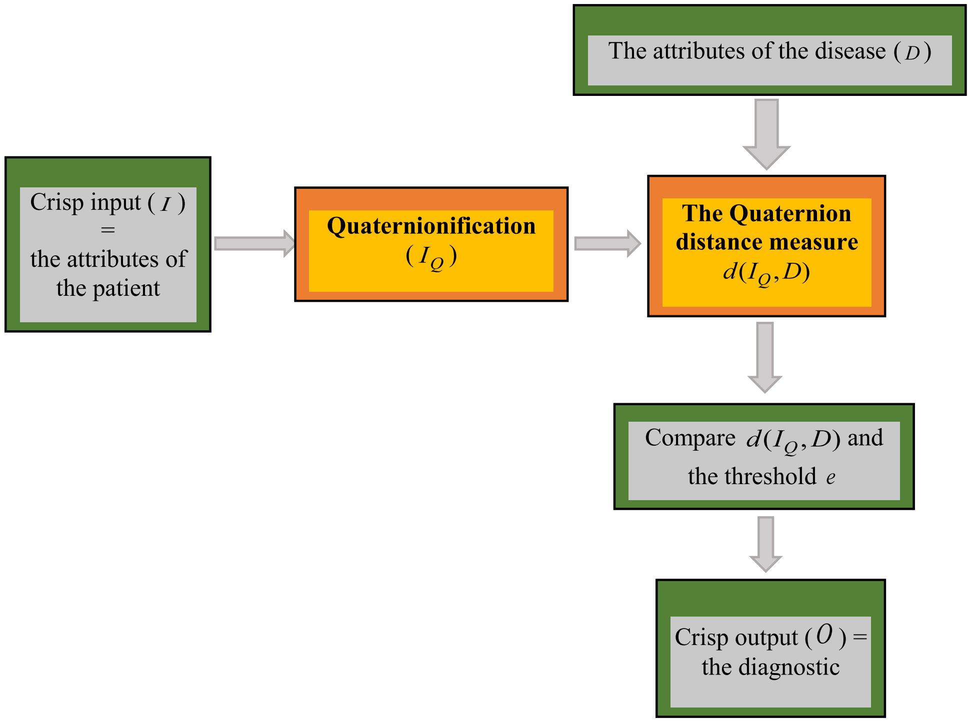

Fig. 5. A diagram of the QDM model.

$$\dddot{\omega}_l(x) = \begin{cases} 1 & \text{if } x < a'_l \\ \left(\frac{b'_l - x}{b'_l - a'_l}\right)^2 & \text{if } a'_l \le x \le b'_l \\ 0 & \text{if } x > b'_l \end{cases}, \tag{49}$$

and

$$\dddot{\gamma}_l(x) = \begin{cases} 0 & \text{if } x < a_l \\ \left(\frac{x - a_l}{b_l - a_l}\right)^2 & \text{if } a_l \le x \le b_l \\ 1 & \text{if } x > b_l \end{cases}, \tag{50}$$

where $a_l \ge a'_l, b_l \ge b'_l$. Here, the quaternionification functions are built based on the traditional fuzzification function, named L-function [46], and conditions (8)–(12) of Definition 6. We obtain quaternions in the Cartesian form of the values p_{il} as,

$$\left(\dddot{\alpha}_l(p_{il}), \dddot{\beta}_l(p_{il}), \dddot{\omega}_l(p_{il}), \dddot{\gamma}_l(p_{il})\right) = \left(\dddot{\alpha}_{il}, \dddot{\beta}_{il}, \dddot{\omega}_{il}, \dddot{\gamma}_{il}\right). \tag{51}$$

The final column of Table 1 presents the real values $Y = \{y_i\}_{i=1,\dots,m}$, which are the diagnosis classification results of patients.

2. *Training Process* : Assuming that the training dataset is a set of t records (see Table 1), and the testing includes $(m-t)$ remaining records (by splitting the original dataset by the 4-Fold cross-validation). Inspired by using the Pearson correlation coefficient function between S_l and Y (see Table 1) to build a medical knowledge base in linguistic approach by Phong et al. [47], here we obtain the attributes characteristic for the disease D:

$$\left(\dddot{\alpha}_{ld}, \dddot{\beta}_{ld}, \dddot{\omega}_{ld}, \dddot{\gamma}_{ld}\right), l = 1, \dots, n, \tag{52}$$

where

$$\dddot{\alpha}_{ld} = \frac{\left|M[\dddot{\alpha}_{il}Y] - M[\dddot{\alpha}_{il}]M[Y]\right|}{\sqrt{M[\dddot{\alpha}_{il}^2] - M[\dddot{\alpha}_{il}]^2}\sqrt{M[Y^2] - M[Y]^2}}, \tag{53}$$

$$\dddot{\beta}_{ld} = \min\left\{1 - \dddot{\alpha}_{ld}, \frac{\left|M[\dddot{\beta}_{il}Y] - M[\dddot{\beta}_{il}]E[Y]\right|}{\sqrt{M[\dddot{\beta}_{il}^2] - M[\dddot{\beta}_{il}]^2}\sqrt{M[Y^2] - M[Y]^2}}\right\}, \tag{54}$$

$$\dddot{\omega}_{ld} = \min\left\{1 - \dddot{\alpha}_{ld}, \frac{\left|M[\dddot{\omega}_{il}Y] - M[\dddot{\omega}_{il}]M[Y]\right|}{\sqrt{M[\dddot{\omega}_{il}^2] - M[\dddot{\omega}_{il}]^2}\sqrt{M[Y^2] - M[Y]^2}}\right\}, \tag{55}$$

and

$$\dddot{\gamma}_{ld} = \min\left\{1 - \dddot{\beta}_{ld}, 1 - \dddot{\omega}_{ld}, \frac{\left|M[\dddot{\gamma}_{il}Y] - M[\dddot{\gamma}_{il}]M[Y]\right|}{\sqrt{M[\dddot{\gamma}_{il}^2] - M[\dddot{\gamma}_{il}]^2}\sqrt{M[Y^2] - M[Y]^2}}\right\}, \tag{56}$$

where $M\left[X = \{x_i\}_{i=1,\dots,t}\right] = \frac{1}{t}\sum_{i=1}^{t} x_i, \ i = 1, \dots, t.$

3. *Testing Process* : The relation between patient p_i $(i = (t+1), \dots, m)$ and the disease D is calculated via the measure in formula (35):

$$d_i = d_E(p_i, D) = \frac{1}{4n}\sum_{l=1}^{n}\sqrt{(\dddot{\alpha}_{il} - \dddot{\alpha}_{ld})^2 + (\dddot{\beta}_{il} - \dddot{\beta}_{ld})^2 + (\dddot{\omega}_{il} - \dddot{\omega}_{ld})^2 + (\dddot{\gamma}_{il} - \dddot{\gamma}_{ld})^2}. \tag{57}$$

Hence, if the set of disease level labels is $\{1, 2\}$, then the diagnostic of the patients is defined as follows:

$$y_i = \begin{cases} 1 & \text{if } d_i \ge e \\ 2 & \text{if } d_i < e \end{cases}, \forall i = (t+1), \dots, m, \tag{58}$$

where $e \in [0, 1]$ is the optimal trained threshold from the testing process for t records used for training process.

Remark 6. In order to simplify the proposed algorithm, the sets of parameters in the Quaternionification step are chosen by a medical expert and the parameter e in (58) is optimized on [0, 1]. Specifically, the optimization problem here is stated as follows: $f(e) = \frac{1}{t}\sum_{i=1}^{t}\left|y_i(e) - y_i^*\right| \to min,\ 0 \le e \le 1$, where

$$y_i(e) = \begin{cases} 1 & \text{if } d_i \ge e \\ 2 & \text{if } d_i < e \end{cases}$$

and $y_i^* \in \{1; 2\}$ is the observed diagnostic result from the dataset. We consider a discrete objective function $f(e)$ with e being rounded to k decimals for a given fixed number k. Hence, this problem takes into the form of a discrete optimization problem with $e \in \left\{0, \frac{1}{10^k}, \frac{2}{10^k}, \ldots, \frac{10^k-1}{10^k}, 1\right\}$. A brute-force approach would evaluate the functions for each of the 10^k values of e to obtain the optimal solution.

Example 4. We consider four records referenced from the data used in [7] (see Table 2).

In the column Class of Viral Fever of Table 2, Label 1 means that the patient suffers from Viral Fever and Label 2 means that the patient does not suffer from Viral Fever. The final row shows the relations of the attributes and the Viral Fever disease.

The degree of the relation between each patient p_i $(i = 1, \ldots, 4)$ and the Viral Fever disease is calculated by the Euclidean quaternion distance measure given by formula (57). Hence, we have:

$$d_E(p_1, \text{Viral fever}) = \frac{1}{20}(\sqrt{0.27} + \sqrt{0.5} + \sqrt{0.07} + \sqrt{0.17} + \sqrt{0.06}) = 0.107.$$

Similarly, we obtain $d_E(p_2, \text{Viral fever}) = 0.159$, $d_E(p_3, \text{Viral fever}) = 0.124$, and $d_E(p_4, \text{Viral fever}) = 0.088$.

Here, the class of Viral Fever of the patients is defined as follows:

$$y_i = \begin{cases} 1 & \text{if } d_E(p_i, D) \ge 0.1 \\ 2 & \text{if } d_E(p_i, D) < 0.1 \end{cases}, \forall i = 1, \ldots, 4.$$

Thus, we conclude: the patients p_4 suffer from Viral Fever, while the patients p_1, p_2, and p_3 do not. This diagnostic determination is the same as the result of Class of Viral Fever in Table 2.

- **P-QDM Method**

P-QDM is built by replacing the Cartesian representation and the Euclidean quaternion distance measure of the method C-QDM with the polar representation and the θ-distance measure. Specifically, the replacement is as follows.

(1) In the step *Quaternionification*: Similar to the C-QDM, the quaternionification functions in the polar form are also based on the traditional fuzzification functions, such as the triangular, trapezoidal, Gaussian functions, L-function, and R-function [46]. Specifically, the relations between patient p_i and attributes S_l are represented in the polar form by the modulus and argument functions:

$$\left([[\ddot{r}_{\mu il}, \theta_{\mu il}]], [[\ddot{r}_{\nu il}, \theta_{\nu il}]]\right) = \left([[\ddot{r}_{\mu l}(p_{il}), \theta_{\mu l}(p_{il})]], [[\ddot{r}_{\nu l}(p_{il}), \theta_{\nu l}(p_{il})]]\right), \tag{59}$$

where

$$\ddot{r}_{\mu l}(x) = \begin{cases} 1 & \text{if } x < a_l \\ \left(\frac{b_l - x}{b_l - a_l}\right)^2 & \text{if } a_l \le x \le b_l \\ 0 & \text{if } x > b_l \end{cases}, \tag{60}$$

$$\theta_{\mu l}(x) = \begin{cases} \frac{\pi}{2} & \text{if } x < a_l \\ \frac{\pi}{2}\cdot\frac{b_l - x}{b_l - a_l} & \text{if } a_l \le x \le b_l \\ 0 & \text{if } x > b_l \end{cases}, \tag{61}$$

$$\ddot{r}_{\nu l}(x) = \begin{cases} 0 & \text{if } x < a_l \\ \left(\frac{x - a_l}{b_l - a_l}\right)^2 & \text{if } a_l \le x \le b_l \\ 1 & \text{if } x > b_l \end{cases}, \tag{62}$$

and

$$\theta_{\nu l}(x) = \begin{cases} 0 & \text{if } x < a_l \\ \frac{\pi}{2}\cdot\frac{a_l - x}{b_l - a_l} & \text{if } a_l \le x \le b_l \\ -\frac{\pi}{2} & \text{if } x > b_l \end{cases}, \tag{63}$$

where $i = 1, \ldots, m$; $l = 1, \ldots, n$.

(2) In the step *Training Process*: By the correlation relationship between S_l and Y (see Table 1), the attributes characteristic for the disease D are represented by

$$\left([[\ddot{r}_{\mu ld}, \theta_{\mu ld}]], [[\ddot{r}_{\nu ld}, \theta_{\nu ld}]]\right), l = 1, \ldots, n, \tag{64}$$

where

$$\theta_{\mu ld} = \frac{\left|M[\theta_{\mu il}Y] - M[\theta_{\mu il}]M[Y]\right|}{\sqrt{M[\theta_{\mu il}^2] - M[\theta_{\mu il}]^2}\sqrt{M[Y^2] - M[Y]^2}}, \tag{65}$$

$$\ddot{r}_{\mu ld} = \min\left\{\frac{1}{\sqrt{2}\cos(\frac{\pi}{4} - \theta_{\mu ld})}, \frac{\left|M[\ddot{r}_{\mu il}Y] - M[\ddot{r}_{\mu il}]M[Y]\right|}{\sqrt{M[\ddot{r}_{\mu il}^2] - M[\ddot{r}_{\mu il}]^2}\sqrt{M[Y^2] - M[Y]^2}}\right\}, \tag{66}$$

$$\theta_{\nu ld} = \left|\frac{M[\theta_{\nu il}Y] - M[\theta_{\nu il}]M[Y]}{\sqrt{M[\theta_{\nu il}^2] - M[\theta_{\nu il}]^2}\sqrt{M[Y^2] - M[Y]^2}}\right|, \tag{67}$$

and

$$\ddot{r}_{\nu ld} = \min\left\{1 - \ddot{r}_{\mu ld}, \frac{1}{\sqrt{2}\cos(\frac{\pi}{4} + \theta_{\nu ld})}, \frac{|M[\ddot{r}_{\nu il}Y] - M[\ddot{r}_{\nu il}]M[Y]|}{\sqrt{M[\ddot{r}_{\nu il}^2] - M[\ddot{r}_{\nu il}]^2}\sqrt{M[Y^2] - M[Y]^2}}\right\}, \tag{68}$$

where $M\left[X = \{x_i\}_{i=1,\ldots,t}\right] = \frac{1}{t}\sum_{i=1}^{t} x_i,\ i = 1, \ldots, t$.

(3) In the step *Testing Process*: we use the measure defined by (44)–(46):

$$d_\theta(p_i, D) = \frac{1}{2n}\sum_{l=1}^{n}\left(d_{1\theta}(\ddot{\mu}_{il}, \ddot{\mu}_{ld}) + d_{2\theta}(\ddot{\nu}_{il}, \ddot{\nu}_{ld})\right), \tag{69}$$

where

$$d_{1\theta}(\ddot{\mu}_{il}, \ddot{\mu}_{ld}) = \begin{cases} 1 - \cos\left(\frac{1}{2}\left|\theta_{\mu il} - \theta_{\mu ld}\right| + \frac{\pi}{4}\right) & \text{if } \theta_{\mu il} \ne \theta_{\mu ld} \\ \left|\ddot{r}_{\mu il} - \ddot{r}_{\mu ld}\right|\left(1 - \cos\left|\theta_\mu - \frac{\pi}{4}\right|\right) & \text{if } \theta_{\mu il} = \theta_{\mu ld} = \theta_\mu \end{cases}, \tag{70}$$

$$d_{2\theta}(\ddot{\nu}_{il}, \ddot{\nu}_{ld}) = \begin{cases} 1 - \cos\left(\frac{1}{2}\left|\theta_{\nu il} - \theta_{\nu ld}\right| + \frac{\pi}{4}\right) & \text{if } \theta_{\nu il} \ne \theta_{\nu ld} \\ \left|\ddot{r}_{\nu il} - \ddot{r}_{\nu ld}\right|\left(1 - \cos\left|\theta_\nu + \frac{\pi}{4}\right|\right) & \text{if } \theta_{\nu il} = \theta_{\nu ld} = \theta_\nu \end{cases}. \tag{71}$$

Table 2
Records on Viral Fever disease.

No.	Temperature	Headache	Stomach pain	Cough	Chest pain	Class of Viral Fever
1	(0.8,0,0.1,0.6)	(0.6,0,0.1,0.5)	(0.2,0.7,0.8,0.1)	(0.6,0,0.1,0.5)	(0.1,0.7,0.6,0.1)	1
2	(0,0.7,0.8,0.1)	(0.4,0.2,0.4,0.3)	(0.6,0,0.1,0.5)	(0.1,0.5,0.7,0.2)	(0.1,0.7,0.8,0.2)	1
3	(0.8,0,0.1,0.6)	(0.8,0,0.1,0.6)	(0,0.4,0.6,0.2)	(0.2,0.6,0.7,0.1)	(0,0.4,0.5,0.1)	1
4	(0.6,0,0.1,0.5)	(0.5,0.3,0.4,0.4)	(0.3,0.5,0.4,0.1)	(0.7,0.1,0.2,0.6)	(0.3,0.5,0.4,0.1)	2
Viral fever	(0.4,0.1,0,0.3)	(0.3,0.4,0.5,0.2)	(0.1,0.5,0.7,0.2)	(0.4,0.3,0.3,0.5)	(0.1,0.5,0.7,0.2)	

Table 3
Description of the benchmark datasets.

Datasets	Number of classes	Number of attributes	Number of elements
ILPD	2	8	583
Diabetes	2	4	389
HSD	2	3	306
E. coli	2	5	336
BCWD	2	9	683

Table 4
Mean absolute error (MAE).

	ILPD	Diabetes	HSD	E. coli	BCWD
SM1-1	0.3195	0.1581	0.2515	0.2303	0.2196
SM1-2	0.3158	0.1612	0.2513	0.2319	0.6494
SM1-3	0.3316	0.1593	0.263	0.2293	0.325
SM1-4	0.2918	0.1609	0.2532	0.2332	0.6497
SM2	0.2902	0.1597	0.2639	0.2317	0.5257
WXM	0.3227	0.1638	0.2513	0.2422	0.2957
VSM	0.2893	0.1634	0.2515	0.2301	0.6493
ZJM	0.3096	0.1579	0.2518	0.2327	0.1226
WM	0.2915	n.a.	0.2635	n.a.	n.a.
JM	0.289	n.a.	0.26	n.a.	n.a.
SAM	0.3031	n.a.	0.2822	n.a.	n.a.
H-max	0.2848	0.1497	0.2476	0.2258	0.1398
C-QDM	0.2836	0.0763	0.2411	0.2162	0.0248
P-QDM	0.2831	0.1125	0.2421	0.2249	0.0316

7. Experiments

7.1. Experimental environments

We compare the proposed methods (C-QDM, P-QDM) to the methods of Wang & Xin [36] (WXM), Szmidt & Kacprzyk [35] (SM1-1, SM1-2, SM1-3, SM1-4), Szmidt & Kacprzyk [23] (SM2), Vlachos & Sergiadis [20] (VSM), Zhang & Jiang [37] (ZJM), Wei et al. [38] & Hung [39] (WM), Maheshwari & Srivastava [41] (SAM), Jujun et al. [40] (JM), and Ngan et al. [7] (H-max). The comparison is done using MATLAB (2015a) and the R programming language. The algorithms contain three main parts:

i. Determining and encoding relations between patients and symptoms.
ii. Formulating and encoding relations between symptoms and diagnoses; thus constructing a medical knowledge base.
iii. Determining the diagnoses based on calculating the distance between the above two relations.

The contributions of the proposed algorithms (C-QDM, P-QDM) are as follows. They use the quaternionification process instead of the fuzzification process. Thus, they add one more dimension of fuzziness to complex intuitionistic fuzzy sets. C-QDM and P-QDM use new distance measures, the Euclidean quaternion distance measure and the θ-distance measure, which replace the previous distance measures.

For the present comparative analysis, we have utilized five benchmark datasets (see Table 3): Indian Liver Patient Dataset (ILPD), Haberman's Survival Dataset (HSD), E. coli Dataset (E. coli), and Breast Cancer Wisconsin (Original) Dataset (BCWD) from the University of California (UCI) Machine Learning Repository (https://archive.ics.uci.edu/ml/index.php); the Diabetes dataset from the Department of Biostatistics, at Vanderbilt University (http://biostat.mc.vanderbilt.edu/wiki/Main/DataSets). Note that some of the distance measure based decision-making methods do not work with some datasets. The reason is that distance measure functions may be undetermined at some values of the datasets.

7.2. Experimental results

We obtain the experimental results of fourteen methods as in Tables 4–5. Table 4 presents the average MAE values of C-QDM and P-QDM against the related methods denoted by SM1, SM2, WXM, VSM, ZJM, WM, JM, SAM, and H-max on the ILPD, Diabetes, HSD, E. coli, and BCWD datasets. It can be observed that the MAE values of C-QDM and P-QDM are less (better) than those of the other algorithms on the five considered datasets. On the ILPD dataset, 0.2836 and 0.2831 are less than all values of 0.3195, 0.3158, 0.3316, 0.2918, 0.2902, 0.3227, 0.2893, 0.3096, 0.2915, 0.289, 0.3031, and 0.2848 respectively.

Table 5 presents the computational time in seconds (sec) of C-QDM and P-QDM against the related algorithms on ILPD, Diabetes, HSD, E. coli, and BCWD. For example in Table 5, the computational times of the C-QDM algorithm is the best (smallest) value, which is 0.155 s (sec), on the dataset of ILPD. In general, the computational time of the C-QDM and P-QDM are not much different from those of the other methods. In some cases, the computation time of the proposed methods is longer, but that is not a determining factor in this type of problems. We note that the computation time of our algorithms is very close to the computation time of the other methods. However, the errors of the C-QDM and P-QDM are significantly lower than those of the other algorithms on some datasets and this is a determining factor that would justify the minor additional computational time. For example, C-QDM runs in 0.1724 s on the dataset of BCWD while the H-max and WXM algorithms run in 0.157 and 0.1636 s, respectively. On the other hand, the MAE value of C-QDM on the dataset of BCWD, which is 0.0248, is significantly better than those of H-max and WXM, which are 0.1398 and 0.2957, respectively.

In Tables 4 and 5, the methods WM, JM, and SAM are labeled as "n.a". on the Diabetes dataset. That is, they do not produce output values due to the measures used in the algorithms WM, JM, and SAM are not determined by values of the Diabetes dataset. Therefore, comparing it with the proposed method and others, WM, JM and SAM are deficient as they cannot work on such datasets.

The obtained MAE values of all of the methods are depicted to provide visualization of the above findings. Fig. 6 shows the values of MAE of the 14 methods on ILPD. In the figure, the heights of the vertical bars show the MAE values of the corresponding algorithms. The heights of the red and green bars are lower than those of 12 remaining bars. Hence, the MAE values of C-QDM and P-QDM are better than those of other methods on ILPD

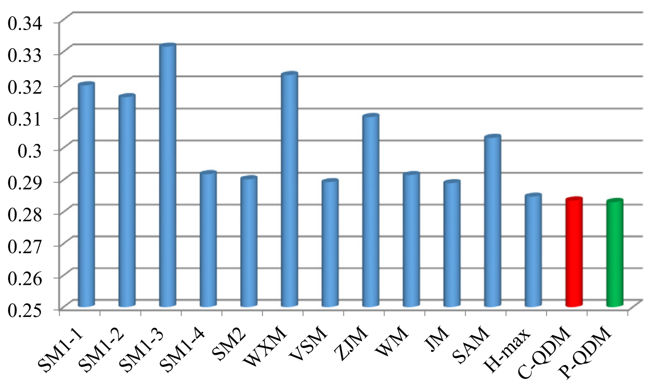

Fig. 6. The MAE results on ILPD.

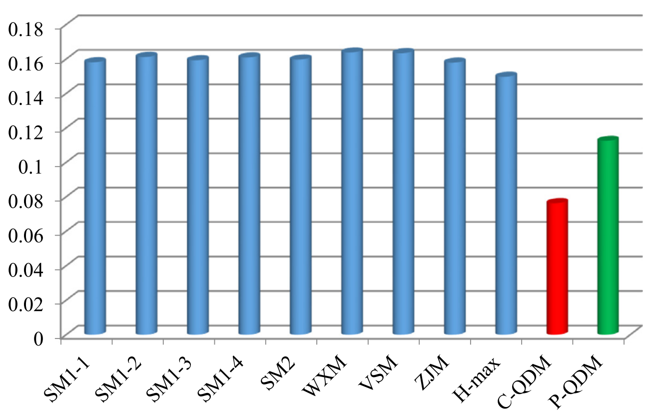

Fig. 7. The MAE results on Diabetes.

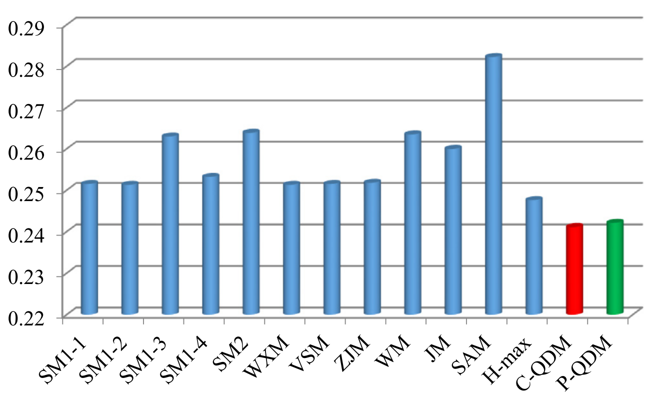

Fig. 8. The MAE results on HSD.

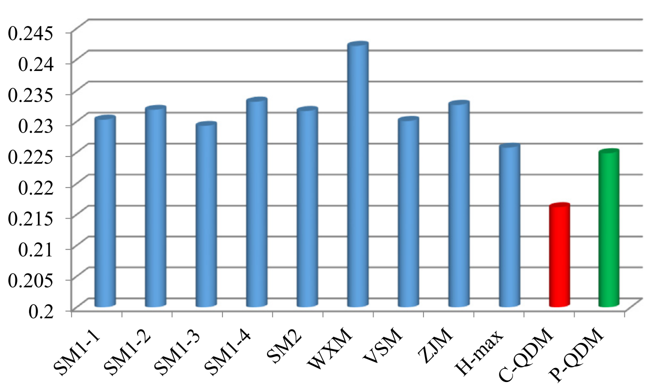

Fig. 9. The MAE results on E. coli.

Table 5
Total time (Sec.).

	ILPD	Diabetes	HSD	E. coli	BCWD
SM1-1	0.6177	0.3237	0.027	0.13	0.122
SM1-2	0.4427	0.2737	0.036	0.14	0.162
SM1-3	0.4827	0.3062	0.077	0.14	0.177
SM1-4	0.4602	0.3012	0.087	0.145	0.197
SM2	0.6527	0.3087	0.092	0.17	0.1945
WXM	0.4427	0.3562	0.082	0.145	0.1636
VSM	0.5552	0.3637	0.147	0.24	0.412
ZJM	0.5602	0.3737	0.177	0.265	0.372
WM	0.8452	n.a.	0.202	n.a.	n.a.
JM	1.2077	n.a.	0.217	n.a.	n.a.
SAM	0.8102	n.a.	0.107	n.a.	n.a.
H-max	0.51	0.3062	0.034	0.1	0.157
C-QDM	0.155	0.1114	0.0326	0.151	0.1724
P-QDM	0.469	0.2614	0.1505	0.1939	0.246

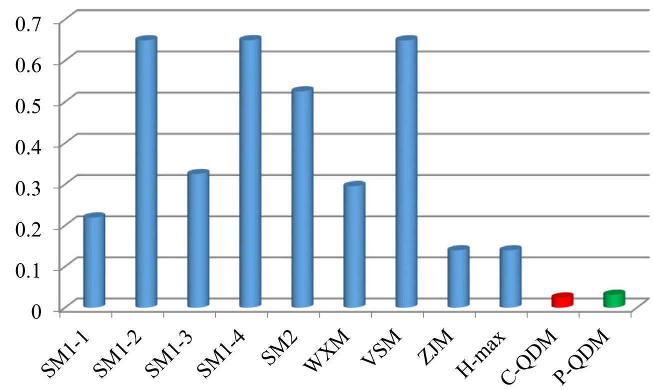

Fig. 10. The MAE results on BCWD.

The bars in Figs. 7–10 depict the average MAE values in similarity to Fig. 6. It can be observed that the MAE of C-QDM corresponding to the height of the red bar is the best on Diabetes, HSD, E. coli, and BCWD. The accuracy of C-QDM and P-QDM on Diabetes are high, which are about 93% and 89% (see Fig. 7). Specifically, C-QDM and P-QDM are effective on the BCWD dataset, where they achieve over 95% accuracy (see Fig. 10), while the H-max method proposed in 2018 achieves about 86%.

8. Conclusion

A new concept of IFSs based on quaternion numbers was introduced in this paper. In this way, we maintain four degrees of freedom, which are the degrees of real membership, imaginary membership, real non-membership, and imaginary non-membership. Several operations have been defined and their properties studied. Further, a new intuitionistic fuzzy order relation and distance measures based on quaternion numbers were proposed. These theoretical methods provide a generalized way of analyzing vague information.

The applicability of the proposed approach is certified by presenting a new Quaternion Distance Measure decision-making model, referred to as QDM. It has been noted that fuzzification in decision-making approach using IFSs is an important process. In order to obtain better accuracy in decision-making, we have proposed expanding the fuzzification process by quaternionification. The QDM model was experimentally tested on medical diagnosis benchmark data and compared with twelve related methods. The results obtained in the tests are better than for the above-listed methods. The computational time of the QDM is not much different from the others.

Future research stemming from the present work can explore other quaternionification functions and improve the parameters of quaternionification in decision making. Further, expanding the uses of quaternion number representation to represent complex intuitionistic fuzzy set theory and logic can be considered. Another future research direction is further study of complex intuitionistic fuzzy linguistic logic in modeling multiple criteria decision making. Additionally, a combination of linear programming methods and complex intuitionistic fuzzy linguistic sets will be investigated. The foregoing representation of IFSs in fuzzy neural networks will be further studied.

Declaration of competing interest

No author associated with this paper has disclosed any potential or pertinent conflicts which may be perceived to have impending conflict with this work. For full disclosure statements refer to https://doi.org/10.1016/j.asoc.2019.105961.

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Appendix

The link https://sourceforge.net/projects/qdmcode/ provides the source code of software and datasets used in and developed for this paper.

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