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Logic Connectives of Complex Fuzzy Sets

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Abstract. The Fuzzy Set Theory has been applied in various problems in numerous fields. In particular, the concepts of t-norms and t-conorms serve a significant role in shaping the theory and its applications. The notion of Complex Fuzzy Sets extends the Fuzzy Set Theory and provides several advantages over the classical theory, especially in terms of the capability to concisely, efficiently, and accurately represent complex relations between fuzzy set components. Some of the areas where complex fuzzy sets have been successfully applied are the areas of time series analysis and multi-criteria decision making problems. The notions of complex t-norms and t-conorms have not been fully developed so far. In this paper, we present the complex fuzzy set forms of t-norms and t-conorms and detail their properties. Additionally, we provide two numerical examples of applying the complex t-norm and t-conorm to multi-criteria decision making in the context of medicine- related problems using medical datasets.

Key-words: Fuzzy sets, Complex fuzzy sets, t-norm and t-conorm, Complex fuzzy t-norms and t-conorms, Multi-criteria decision making problem.

1. Introduction

Fuzzy Logic, along with fuzzy sets, introduced by Zadeh in 1965 [34], is a continuous multi-valued logic system. Numerous extensions to the theory of fuzzy sets and fuzzy logic have been introduced. These extensions include Type-2 fuzzy sets [19], neutrosophic fuzzy sets [29], intuitionistic fuzzy sets [4], picture fuzzy sets [9], and complex fuzzy sets [26]. Applications of these extensions are in areas such as control theory [23, 36, 14] and artificial intelligence [28].

An important extension to the fuzzy set and logic theory, namely complex fuzzy sets and complex fuzzy logic, has been investigated by Ramot et al. [26]. Tamir et al. rigorously reintroduced basic concepts of complex fuzzy sets (CFS), complex fuzzy logic (CFL), complex fuzzy inference, and generalized complex fuzzy classes [30]. A survey of the applications related to complex fuzzy logic and complex fuzzy sets was also presented in the latter paper. Ali et al. have proposed new systems for CFS and CFL [1]. However, they used the traditional max and min operations as a representation of t-norms and t-conorms.

Yazdanbakhsh et al. presented a systematic review of CFS and CFL [33]. Their research has concentrated on uncertainty representation in CFS and CFL operations and relations. However, t-norms and t-conorms in CFS are not discussed in the paper.

One of the key concepts of fuzzy set theory is the concept of Triangular Norms (t-norms), Triangle conorms (t-conorms), and negation functions, which are used to calculate the membership values of fuzzy-sets intersection, union and complement, respectively. The name “triangular norm” refers to the fact that in the framework of probabilistic metric spaces, t-norms and t-conorms are used to generalize triangle inequality of ordinary metric spaces [12]. Triangular norm-based fuzzy logic is a family of non-classical logic functions, informally described by having a semantics that maps the real unit interval [0, 1] to the system of truth values. Triangular norms constitute a set of binary operation used in the framework of probabilistic metric spaces and in multi-valued logic; specifically, in fuzzy logic. Triangular norms and conorms are operations which generalize the logical conjunction and logical disjunction to fuzzy logic. They are a natural interpretation of the conjunction and disjunction in the semantics of mathematical fuzzy logic. They are mainly used in applied fuzzy logic and fuzzy set theory as a basis for approximate reasoning, where they can generate a well-formed form of implication and they are often used to combine criteria in multi-criteria decision making (MCDM) [32].

Upon extensive review of the state of the art in complex fuzzy sets and complex fuzzy logic, we have concluded that there is a need for a rigorous definition of complex fuzzy logic t-norms. A focus of the present paper is to propose extensions of the concepts of t-norm and t-conorm equations for complex fuzzy sets.

In this paper, we propose a notation for and definitions of complex t-norm and t-conorm with properties that extend the classical definition. Aggregation operators based on these t-norm and t-conorm are also presented. Additionally, we illustrate application of the obtained results via a numerical example using the Tsunomoto dataset [15] and a real-world case at Thai Nguyen Hospital.

The remaining of this paper is organized in the following way: Section 2 provides a broad literature review. Section 3 presents several basic concepts of fuzzy sets and complex fuzzy sets, as well as their operations and relations. Section 4 introduces the extension to complex fuzzy sets of t-norms and t-conorms, referred to as complex t-norms and t-conorms, respectively. Section 4 also discusses the exponential operational law and an aggregation technique for complex fuzzy numbers (EOL-CFN). Numerical examples and related observations are presented in Section 5. Finally, our conclusions are stated in Section 6.

2. Literature Review

In this section we provide broad literature review on important related concepts. We begin with discussion of recent literature related to complex fuzzy and its applications as well as to the theory and applications of evolution equations. Next, we survey current research on multi-criteria decision making (MCDM). Finally, we introduce and discuss concepts of triangular norms.

Kandel *et al.* have used complex fuzzy sets in multi-dimension spaces in order to solve disaster mitigation and management problem [17]. Commonly used techniques utilized in decision making systems based on fuzzy sets and CFS are detailed in that paper. Additionally, its authors have evaluated the strength and the weakness of different fuzzy techniques. Specifically, they have addressed one of the most challenging issues related to disaster mitigation, where slowly evolving uncertain data is becoming available as the disaster is progressing. CFS applications in time series forecasting and short-term forecasts include the ANCFIS [8], ARIMA [22], and CNFS [21]. These systems yield excellent nonlinear mapping capability for time-series.

Our present paper demonstrates the utility of t-norms and t-conorms in MCDM systems. Hence, in the following we survey important related systems and research.

Fuzzy inference is widely used in decision making support systems. For example, fuzzy inference is applied in human resource management in [16, 25]. In the latter paper, a fuzzy method based on Mamdani fuzzy inference system (FIS) was proposed to assist managers in decision making and to obtain high performance in employment decisions. Casanova has presented a hybrid intelligent system based on a decision support system and a fuzzy inference system [7]. His paper is an example of the application of FIS to financial analysis and his system has been applied in stock selection and portfolio management. Outputs of the latter system have been compared with actual stock performance, demonstrating that investors could make significant profit if buying stocks based on the system's recommendations. In a related research, Supriya et al [11] have presented an application of the k-nearest neighbors, thereby providing a new approach to data mining in fuzzy databases.

Recently, Gupta pointed out essential applications of fuzzy sets and fuzzy logic in real life applications such as transportation, trip distribution, traffic signal control, decision and investment, washing machines, air chillers, and in medical applications [12, 13, 18]. Many of these applications use systems which utilize the concepts of t-norm and t-conorms; these operators are discussed next.

The theory of evolution equations has become an important area of investigation in recent years, stimulated by their applications to numerous problems in the areas of mechanics, medicine, biology, ecology, and others. Under this theory, it is assumed that real-world information is not collected all at once or at the same time. Specifically, for many realistic complex problems, such as medical diagnosis problems, decision-making problems, and business investment decisions, the required information frequently needs to be collected and accumulated from numerous resources at different times. In such cases, complex fuzzy set theory can become an important basis for addressing these complex evolution equations-based systems.

A t-norm generalizes the intersection operator in the set theory and the conjunction operator in logic. As members of the family of fuzzy logic, t-norm operations aim at generalizing the classical two-valued logic by admitting intermediate truth values between 0 and 1, representing degrees of truth of propositions. The degrees are assumed to be real numbers within the unit interval [0, 1] [12]. On the other hand, t-conorms allow to evaluate the truth degrees of compound formulas. They are applied in fuzzy control to formulate assumptions of rules as a conjunction of fuzzy sets, called antecedents or premises. The *minimum or product t*-norms are typically used in these applications because of a lack of motivation for other t-norms [10].

Various terms, along with Zadeh's conventional t-operators, MIN and MAX, have been used in numerous designs of fuzzy logic controllers and in the modelling of decision-making processes. Nevertheless, theoretical and experimental studies seem to indicate that other types of t-operators may work better in specific situations, especially in the context of decision-making processes. For example, the product operator may be preferred to the MIN operator [27]. On the other hand, when choosing a set of T-operators for a given decision-making process, one has to consider their properties, the accuracy of the model, their simplicity, software and hardware implementation, etc.

For these and other reasons, it is of interest to consider different sets of t-operators in the modelling of decision-making processes, so that multiple options be available for selecting t-operators that may be better suited for given problems. Furthermore, the subject of t-operators in the context of CFS and CFL has not been thoroughly investigated. The present paper is attempting to overcome this deficiency.

The Lukasiewicz t-conorm is closely related to the basic binary operation of multi-valued algebras. Additionally, t-norms and t-conorms form examples of aggregation operators. They play a significant role in the axiomatic definition of the concept of triangular norm-based measure and, in particular, of the concept of probability of fuzzy events; the Frank family of t-norms and t-conorms plays a particular role [6].

Gupta *et al.* provide a detailed review of t-norm and t-conorm [12]. The operators t-norm and t-conorm were considered as a part of T-operators. In that paper, the authors define T-operators and their properties. They also propose a fuzzy reasoning method based on min and max operators. This method has been applied to model a decision making support system.

Alsina et al. [2] have introduced the t-norm and the t-conorm into fuzzy set theory and suggested that t-norm and t-conorm be used for the intersection and union of fuzzy sets. Based on t-norm and t-conorm concepts, Bodjanova et al. have proposed a new concept of fuzzy sets, namely T-superior [5]. The T-superiority measure of fuzzy sets is computed via a measure of t-norm. Additionally, their paper presents an application of t-norm and t-conorm in analyzing vague data by introducing the notion of t-superiority.

It should be mentioned that t-norms overlap with copulas [3, 24]: commutative associative copulas are t-norms; t-norms which satisfy the 1-Lipschitz condition are copulas. Some families of t-norms are known as families of copulas under different names.

A new t-norm was introduced and applied in Active Learning Method (ALM) by Kiaei et al. [20]. The original operators of ALM were presented, along with the Ink Drop Spread and the Center of Gravity operators, and two basic morphological operators. The obtained results show that new operators have overcome several of the disadvantages of the original operators. The new operators were applied well in ALM. In that paper, aggregations of fuzzy relations using aggregation functions have been considered. This has been performed by determining certain conditions expressed in computational formulas and deploying t-norms and t-conorms properties. Recently, T-operators have been used to combine criteria in MCDM.

3. Basic concepts

Definition 3.1 [34]. Fuzzy Set

A fuzzy set S, defined on a universe of discourse X, is characterized by a membership function $\eta_s(x)$ that assigns a value in [0,1] to every element $x \in X$. A fuzzy set can be represented as

$$S = \{(x, \eta_s(x)) : x \in X\}.$$

Definition 3.2 [26]. Complex Fuzzy Set

A complex fuzzy set S, defined on a universe of discourse X, is characterized by a membership function $\eta_s(x)$ that assigns any element $x \in X$ a complex-valued grade of membership in S. The values $\eta_s(x)$ all lie within the unit circle in the complex plane, and thus have the form $p_s(x) \cdot e^{j \cdot \mu_s(x)}$, where $p_s(x)$ and $\mu_s(x)$ are both real-valued and $p_s(x) \in [0, 1]$. The term $p_s(x)$ is called the amplitude term and $e^{j \cdot \mu_s(x)}$ is called the phase term. A complex fuzzy set can be represented as

$$S = \{(x, \eta_s(x)) : x \in X\}.$$

Definition 3.3 [26]. Complement of a Complex Fuzzy Set

Let S be a complex fuzzy set on X, and $\eta_s(x) = p_s(x) \cdot e^{j \cdot \mu_s(x)}$ be its complex-valued membership function. The complement of S, denoted as $c(S)$ and is specified by the function

$$\eta_{c(s)}(x) = p_{c(s)}(x) \cdot e^{j \cdot \mu_{c(s)}(x)} = (1 - p_s(x)) \cdot e^{j(2\pi - \mu_s(x))}.$$

Definition 3.4 [26]. Union of Complex Fuzzy Sets Let A and B be two complex fuzzy sets on X, and let $\eta_A(x) = r_A(x) \cdot e^{j \cdot \mu_A(x)}$ and $\eta_B(x) = r_B(x) \cdot e^{j \cdot \mu_B(x)}$ be their membership functions respectively. The union of A and B is denoted as $A \cup B$ and is specified by the function

$$\eta_{A \cup B}(x) = r_{A \cup B}(x) \cdot e^{j \cdot \mu_{A \cup B}(x)} = (r_A(x) \vee r_B(x)) \cdot e^{j(\mu_A(x) \vee \mu_B(x))}$$

where $\vee$ denotes the max-operator.

Definition 3.5 [26]. Intersection of Complex Fuzzy Sets

Let A and B be two complex fuzzy sets on X, and let $\eta_A(x) = r_A(x) \cdot e^{j \cdot \mu_A(x)}$ and $\eta_B(x) = r_B(x) \cdot e^{j \cdot \mu_B(x)}$ be their membership functions respectively. The intersection of A and B is denoted as $A \cap B$ and is specified by the function

$$\eta_{A \cap B}(x) = r_{A \cap B}(x) \cdot e^{j \cdot \mu_{A \cap B}(x)} = (r_A(x) \vee r_B(x)) \cdot e^{j(\mu_A(x) \wedge \mu_B(x))}$$

where $\wedge$ denotes the min-operator.

Next, we provide several basic definitions of complex intuitionistic fuzzy sets and list related set theoretic operations using the norm and conorm notation.

Definition 3.6 [35]. Let A and B be two complex fuzzy sets on X, and let $\eta_A(x) = r_A(x) \cdot e^{j \cdot \mu_A(x)}$ and $\eta_B(x) = r_B(x) \cdot e^{j \cdot \mu_B(x)}$ be their membership functions respectively. The complex fuzzy product of A and B is denoted as A ° B and is specified by the function

$$\eta_{A \circ B}(x) = r_{A \circ B}(x) \cdot e^{j \cdot \mu_{A \circ B}(x)} = (r_A(x) \cdot r_B(x+)) \cdot e^{j2\pi\left(\frac{\mu_A(x)}{2\pi} \cdot \frac{\mu_B(x)}{2\pi}\right)}.$$

Definition 3.7 [12, 18, 31] A function $T : [0, 1] \times [0, 1] \times [0, 1] \longrightarrow [0, 1]$ is called a t-norm if it satisfies the following four conditions:

(1) $T(1, x) = x$, for all x;

(2) $T(x, y) = T(y, x)$, for all x and y;

(3) $T(T(x, y), z) = T(x, T(y, z))$, for all x, y and z;

(4) Whenever $x \le x'$ and $y \le y'$, then $T(x, y) \le T(x', y')$.

Definition 3.8 [12, 18, 31]. A function $S : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ is called a t-conorm if it satisfies the following four conditions:

(1) $S(0, x) = x$, for all x;

(2) $S(x, y) = S(y, x)$, for all x and y;

(3) $S(S(x, y), z) = S(x, S(y, z))$, for all x, y and z;

(4) Whenever $x \le x'$ and $y \le y'$, then $S(x, y) \le S(x', y')$.

Definition 3.9 [12, 18, 31]. If the t-norm function $T(x, y)$ is continuous and $T(x, x) < x$ for all $x \in (0, 1)$, then it is called an Archimedean t-norm. If an Archimedean t-norm is strictly increasing with respect to each variable for $x, y \in (0, 1)$, then it is called a strict Archimedean t-norm.

4. Complex fuzzy t-norms and t-conorms

Definition 4.1: Let $\Im$: $[0, 1]\, x\, [0, 1] \to [0, 1]$ be a mapping where [0,1] is the unit complex disk in the set of complex numbers. Then $\Im$ is called a complex T-norm if the following conditions hold for all $p, q, r \in [0, 1]$, where p, q, r are the complex fuzzy membership grade with $p = p_1 e^{j\omega_1}, q = q_1 e^{j\omega_2}, r = r_1 e^{j\omega_3}$

(1) $\Im(p, q) = \Im(q, p)$,

(2) $\Im(p, q) \le \Im(p, r)$, if $q \le r$,

(3) $\Im(\Im(q, r)) = \Im(p, \Im(p, q), r)$,

(4) $\Im(p, 1) = p$.

Definition 4.2: Let $\Im^*$: $[0, 1]\, x\, [0, 1] \to [0, 1]$ be a mapping where [0,1] is the unit complex disk in the set of complex numbers. Then $\Im^*$ is called a complex T-conorm if the following conditions hold for all $p, q, r \in [0, 1]$, where $p = p_1 e^{j\omega_1}, q = q_1 e^{j\omega_2}, r = r_1 e^{j\omega_3}$ are the complex fuzzy membership grade:

(1) $\Im^*(p, q) = \Im^*(q, p)$,

(2) $\Im^*(p, q) \le \Im^*(p, r)$, if $q \le r$,

(3) $\Im^*(\Im^*(q, r)) = \Im^*(\Im^*(p, q), r)$,

(4) $\Im^*(p, 0) = p$.

Definition 4.3. If the complex fuzzy t-norm function $\Im(p, q)$ is continuous and $|\Im(p, p)| < p$ for all $p \in (0, 1)$, then it is called an Archimedean complex fuzzy t-norm. If an Archimedean complex fuzzy t-norm is strictly increasing with respect to each variable for $p, q \in (0, 1)$, then it is called a strict Archimedean complex fuzzy t-norm.

Definition 4.4. If a complex fuzzy t-conorm function $\Im^*(p, q)$ is continuous and $|\Im^*(p, p)| < p$ for all $p \in (0, 1)$ then it is called an Archimedean complex fuzzy t-conorm. If an Archimedean complex fuzzy t-conorm is strictly increasing with respect to each variable $p, q \in (0, 1)$, then it is called a strict Archimedean complex fuzzy t-conorm.

Example 4.5: We extend the fundamental Zadeh's operators to complex T operators as follows:

(1) $\mathfrak{I}(p,q) = \min(p,q)$,

(2) $\mathfrak{I}_1^*(p,q) = \max(p,q)$.

Example 4.6: Some more complex T operators are:

(1) $\mathfrak{I}_2(p,q) = p \cdot q = p_1 \cdot q_1 e^{(\omega_1+\omega_2)}$,

(2) $\mathfrak{I}_2^*(p,q) = p + q - pq$.

Example 4.7: Let $\bar{1} = 1 + 0j$ and let $\bar{0} = 0 + 0j$. The Lukasiewicz complex fuzzy t-norms are:

$$(1) \quad \mathfrak{I}(p,q) = \max(p+q-\bar{1},\bar{0}) = \max(p_1 \cdot e^{j\cdot\omega_1} + q_1 \cdot e^{j\cdot\omega_2} - \bar{1},\bar{0}) \\ = \max\big((p_1+q_1-1)\cdot e^{j\cdot(\omega_1+\omega_2)},0\big) = \big((p_1+q_1-1)\vee 0)\cdot e^{j\cdot(\omega_1+\omega_2)\vee 0}\big),$$

$$(2) \quad \mathfrak{I}^*(p,0) = \min(p_1+q_1,1) = \min(p_1 \cdot e^{j\cdot\omega_1} + q_1 \cdot e^{j\cdot\omega_2},1) \\ = \min\big((p_1+q_1)\cdot e^{j\cdot(\omega_1+\omega_2)},1\big) = \big((p_1+q_1-1)\wedge 0)\cdot e^{j\cdot(\omega_1+\omega_2)\wedge 1}\big).$$

Proposition 4.8: The complex T-norm $\mathfrak{I}$ and T-conorm $\mathfrak{I}^*$ satisfy the following properties:

(1) $\mathfrak{I}(p,\mathfrak{I}^*(q,r)) = \mathfrak{I}^*(\mathfrak{I}(p,q),\mathfrak{I}(p,r))$,

(2) $\mathfrak{I}^*(p,\mathfrak{I}(q,r)) = \mathfrak{I}(\mathfrak{I}^*(p,q),\mathfrak{I}^*(p,r))$.

Proposition 4.9: The complex T-norm $\mathfrak{I}$ and T-conorm $\mathfrak{I}^*$ satisfy the absorption properties:

(1) $\mathfrak{I}(\mathfrak{I}^*(p,q),p) = p$,

(2) $\mathfrak{I}^*(\mathfrak{I}(p,q),p) = p$.

Proposition 4.10: The complex T-norm $\mathfrak{I}$ and T-conorm $\mathfrak{I}^*$ satisfy the idempotency properties:

(1) $\mathfrak{I}(p,q) = p$,

(2) $\mathfrak{I}^*(p,q) = p$.

Theorem 4.11: Let $\mathfrak{I}_1(p,q) = \min(p,q)$ and let $\mathfrak{I}_1^*(p,q) = \max(p,q)$. Then

$$Distribuivity \Rightarrow Absorption \Rightarrow Idempotency \Rightarrow \begin{cases} \mathfrak{I} = \mathfrak{I}_1 \\ \mathfrak{I}^* = \mathfrak{I}_1^* \end{cases}$$

Definition 4.12: Let $\aleph : [0,1]\,x\,[0,1] \to [0,1]$.Then $\aleph$ is called a negation function if

(1) $\aleph(0) = 1, \aleph(1) = 0$,

(2) $\aleph(p) \leq \aleph(q)$ when $p \geq q$.

Definition 4.13: A negation function $\aleph$ is called strict if

(1) $\aleph(p)$ is continuous and

(2) $\aleph(p) < \aleph(q)\ if\ p > q$ for all $p, q \in [0, 1]$.

Definition 4.14: A negation function $\aleph$ is called involutive if

(1) $\aleph(\aleph(p)) = p$ for all $p \in [0, 1]$.

Example 4.15: We extend Zadeh's negation operator to the complex negation operator as follows:

(1) $\aleph(p) = 1 - p$.

Proposition 4.16: The t-norm $\mathfrak{I}$, the t-conorm $\mathfrak{I}^*$, and negation $\aleph$ satisfy the excluded-middle laws. *i.e*.

(1) $\mathfrak{I}(p, \aleph(p)) = 0$,

(2) $\mathfrak{I}^*(p, \aleph(p)) = 1$.

Proposition 4.17: The t-norm $\mathfrak{I}$, the t-conorm $\mathfrak{I}^*$ and negation $\aleph$ satisfy the De Morgan laws *i.e*.

(1) $\aleph(\mathfrak{I}(p, q)) = \mathfrak{I}^*(\aleph(p), \aleph(q))$,

(2) $\aleph(\mathfrak{I}^*(p, q)) = \mathfrak{I}(\aleph(p), \aleph(q))$.

Theorem 4.18: If $\aleph$ is involutive, then Propositions (4.16), (1)-(2) are equivalent.

Theorem 4.19: If $\aleph$ is involutive, then

(1) $\mathfrak{I}(p, q) = \aleph\big(\mathfrak{I}^*(\aleph(p), \aleph(q))\big)$,

(2) $\mathfrak{I}^*(p, q) = \aleph\big(\mathfrak{I}(\aleph(p), \aleph(q))\big)$.

5. A complex fuzzy t-norm and t-conorm-based decision making algorithm

In this section, we apply the t-norm operators to develop a *multi-criteria decision making* (MCDM) algorithm, which consists of the following steps:

Step 1. Consider a MCDM problem where there are m alternatives $\Upsilon(i = 1, 2, \ldots, m)$ and n criteria $\Lambda_k(k = 1,\ 2, \ldots,\ n)$. The decision maker constructs the decision matrix $\Gamma = (\gamma_{ik})_{m\times n}$, where γ_{ik} represents the degree that the decision maker prefers the alternative Υ_i with respect to the criterion Λ_k. The weights of the criteria are expressed as the CFNs $\alpha_k = \left(\mu_{\alpha_k}.e^{j(\omega_{\alpha_k})}\right)$, $(k = 1,\ 2, \ldots,\ n)$, where μ_{α_k} indicates the amplitude function/degree that the decision maker prefers criterion Λ_k, and ω_{α_k} indicates the phase term/degree.

Step 2. Transform the decision matrix $\Gamma = (\gamma_{ik})_{m\times n}$ into the normalized decision matrix D=$(\lambda_{ik})_{m\times n}$, where

$$\lambda_{ik} = \frac{\gamma_{ik}}{\max\limits_{\forall i,k}\gamma_{ik}},\ i = 1, .., \ m; k = 1, \ldots, n$$

Step 3: Utilize the operators in Example 4.7 to compute the Lukasiewicz complex fuzzy t-norms.

Step 4: Sum up the complex membership grades.

Step 5: Consider the highest score as a candidate for the best ranking.

Next, we present examples of applying these steps in two different cases by using various datasets.

Example 5.1:
Data description: This example uses the real dataset of patient records from the Gangthep Hospital and the Thai Nguyen National Hospital. These patients come to the hospital for examination of their liver function with respect to seven indices, including *Aspartate Aminotransferase* (AST), *Alanine Aminotransferase* (ALT), AST/ALT, *Gamma-glutamyl Transferase* (GGT), Albumin, (*Total Bilirubin* (TB), *Direct Bilirubin* (DB), DB/TB. Based on these results, the physician may ask the patient to perform additional examinations aiming to improve the diagnosis. First, here are some criteria that need to be considered in decision-making:
C1: Indices AST and ALT increase and AST is higher than ALT
C2: Albumin decreases while AST and ALT increase
C3: The ratio DB/TB is less than 20%
C4: The ratio DB/TB is within 20-50%
The following tests need to be performed:
E1: HbsAg, HbeAg, and Hepatitis C tests
E2: Liver function tests (*Prothrombin Time* (PT), *Activated Partial Thromboplastin Time* (APTT), and *International Normalized Ratio* (INR) tests)
E3: Hemolysis test.
Based on expert knowledge, the decision matrix of this problem is in Table 5.1.

Table 1. Decision matrix based on data samples

	C1	**C2**	**C3**	**C4**
E1	0.8	0.5	0.2	0.5
E2	0.2	0.8	0.6	0.2
E3	0.7	0.5	0.9	0.3

We fuzzify this matrix using Gauss function, the resultant matrix is as follows:

Table 2. Fuzzified decision matrix

A_1	P_1	A_2	P_2	A_3	P_3	A_4	P_4
0.9727	-6.44E-14	0.9727	0.1798	0.9727	0.0830	0.9997	0
0.9727	0.0899	0.9727	-1.87E-11	0.9920	0.0391	0.9999	0.0014
0.9809	0.0154	0.9727	0.1798	0.9727	-1.85E-06	0.9997	0.0028

where A_i is the amplitude value and P_i is the phase value of criterion C_i, respectively. The weight vector of all criteria is obtained as:

$$((0.5, 0.4), (0.6, 0.3), (0.3, 0.4), (0.2, 0.6))$$

The fuzzified decision matrix is normalized by the formula:

$$A'_i = \frac{A_i}{Max\{A_{i,}\ i = 1,..,n\}},\ P'_i = \frac{P_i}{Max\{P_i, i = 1,..,n\}}\ \ for\ all\ i = 1,..,n.$$

The normalized decision matrix is formed as:

Table 3. The normalized matrix

A_1	P_1	A_2	P_2	A_3	P_3	A_4	P_4
0.9730	0.0000	0.9730	1.0000	0.9730	0.4616	1.0000	0.0000
0.9728	1.0000	0.9728	0.0000	0.9921	0.4349	1.0000	0.0156
0.9812	0.0857	0.9730	1.0000	0.9730	0.0000	1.0000	0.0156

Using the operators in Example 4.7 to compute Lukasiewicz complex fuzzy t-norms and summing up these values, we get the result as:

Table 4. Fuzzy decision matrix by applying Lukasiewicz complex fuzzy t-norms

A_1	P_1	A_2	P_2	A_3	P_3	A_4	P_4
0.9458	0.1952	0.8988	1.6667	0.8986	0.6879	0.9505	0.0077
0.9386	0.6951	0.9188	0.6667	0.9479	0.4441	0.9807	0.0154
0.9541	0.2808	0.8588	1.6667	0.8886	0.2264	0.9531	0.0231

Defuzzifying the obtained matrix, the final decision matrix is as follows.

Table 5. Fuzzy decision matrix by applying Lukasiewicz complex fuzzy t-norms

	C1	C2	C3	C4
E1	**0.9**	0.92	0.7	**0.94**
E2	0.45	**0.97**	0.8	0.8
E3	0.5	0.5	**0.95**	0.89

From this table, the test corresponding to the largest value (bolded values) in each criterion will be chosen.

Example 5.2:

Data description: This example is performed on the Tsumoto dataset [14]. Indices from clinical tests of patients include: AST *Glutamic Oxaloacetic Transaminase* (GOT), ALT *Glutamic Pyruvic Transaminase* (GPT), *Lactate Dehydrogenase* (LDH), *Alkaliphophatase* (ALP), *Ttotal Protein* (TP), *Albumin* (ALB), *Uric Acid* (UA), *Urea Nitrogen* (UN), and *Creatinine* (CRE)). These indices are used in criteria affecting diagnosis.

Based on these indices, we consider several specific conditions:

C1: The values of UN and CRE are high

C2: The value of UA is high

C3: The ALP in blood is higher than usual

Depending on the values of each index, physicians can diagnose patients with probabilities. Some of the resultant diagnoses are:

D1: Renal impairment disease

D2: Gout disease

D3: Liver related diseases

In this problem, the decision matrix is given in Table 5.6 below

Table 6. Decision matrix

	C1	C2	C3
D1	1	0.5	0.3
D2	0.2	1	0.4
D3	0.5	0.3	0.7

Fuzzifying this matrix using Sinusoidal function, the result matrix is:

Table 7. The fuzzy decision matrix using Sinusoidal complex fuzzy membership function

A_1	P_1	A_2	P_2	A_3	P_3
0.9121	0.5000	0.9454	1.5000	0.9454	0.9615
0.9452	0.9999	0.9234	0.5000	0.9648	0.7176
0.9537	0.5856	0.9144	1.5000	0.9414	0.4999

where A$_i$ is the amplitude value and Pi is the phase value of criterion C$_i$, respectively. The weight vector of all criteria is obtained as:

$$((0.7, 0.2), (0.5, 0.4), (0.5, 0.5)).$$

The fuzzified decision matrix is normalized by the formula:

$$A'_i = \frac{A_i}{Max\{A_{i,}\ \ i = 1, .., 3\}},\ P'_i = \frac{P_i}{Max\{P_i, i = 1, .., 3\}}\ \ for\ all\ i = 1, .., 3.$$

The normalized decision matrix is formed as:

Table 8. The normalized matrix

A_1	P_1	A_2	P_2	A_3	P_3
0.9648	0.3333	1.0000	1.0000	1.0000	0.6410
0.9797	1.0000	0.9571	0.5001	1.0000	0.7177
1.0000	0.3904	0.9588	1.0000	0.9871	0.3333

Using the operators in Example 3.7 we compute Lukasiewicz complex fuzzy t-norms.

Table 9. Result matrix obtained by applying Lukasiewicz complex fuzzy t-norms

A_1	P_1	A_2	P_2	A_3	P_3
0.9167	0.7967	0.9416	1.7778	0.9652	1.1252
0.9509	1.1301	0.9188	1.1111	0.9852	0.9627
0.9598	0.8538	0.9094	1.7778	0.9610	0.8176

Defuzzifying the obtained matrix, the final decision matrix is:

Table 10. The final decision matrix

	C1	C2	C3
D1	**0.96**	0.4	0.01
D2	0.2	**0.8**	0.1
D3	0.01	0.1	**0.94**

From this table, the decision with the largest value (bolded values) in each criterion is chosen.

The results of the complex fuzzy t-norm and t-conorm-based decision making algorithm examples have been verified by physicians to be reasonable conclusions for a decision support system. From the examples, it can be observed that due to the exploitation of the expressive power of complex t-norm and t-conorm functions, the algorithm simplifies the process in a way that makes this MCDM problem efficiently tractable.

6. Conclusion

We have introduced the complex fuzzy sets forms of t-norms and t-conorms along with their properties. Additionally, we detailed two numerical examples of applying the complex t-norm and t-conorm to multi-criteria decision making in the context of medical related problems using medical datasets. The examples are based on a complex fuzzy set theory algorithm that utilizes the definitions of complex t-norm and t-conorm.

In the future, we plan to apply the proposed complex fuzzy t-norm and t-conorm to other decision making problems. Additionally, we plan to explore the extension of t-norms to complex intuitionistic fuzzy classes.

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