Computer Technology for Solving Large Scale Matrix Problems

O.D. Joukov and N.D. Rishe

Moscow State University
Leninskie gory, Moscow 119899, Russia
zhukovo@dec1.sinp.msu.su

Abstract. In this paper it is shown a possibility of creating efficient computing technology for solving matrix-vector-based scientific, engineering, economic, and other problems. It is based on using the new class of hypercomplex systems opened by W.F. Hamilton but formed by a procedure which is different from the procedure proposed by his followers. This method allows, for example, to reduce multiplication/summation of two large size matrixes/vectors to multiplication/summation of only two whole numbers and, in general, to use a single real number as an operand instead of matrix/vector. Transforming matrices and vectors into whole numbers can be performed on the basis of parallel massive/multiprocessor systems.

1 Introduction

Matrix-vector processing is known to find a wide usage for solving numerous scientific, engineering, economic, and other problems. In this paper it is proposed an efficient computing method for matrix-vector processing.

Irish world-known mathematician W.F. Hamilton proposed and to-gether with his followers developed the new class of hypercomplex systems and quaternion algebra based on these systems [1]. Later other mathematicions showed the possibility of construction of a class of hypercomplex systems whose presentations have a dimension 2^d $(d = 0, 1, 2, ...,)$ and can be obtained by a special procedure of doubling. In particular, systems of dimension 1 and 2 are the real and complex number systems respectively. Each of such systems is presented as linear algebra.

All of above systems with a dimension more than 4 don't have properties of associativity and commutability. In this paper is presented other method of construction of associative- commutative linear algebra systems based on hypercomplex systems which are formed by a special procedure of doubling and differed from the above mentioned procedure.

On the basis of this method the sequence of such algebra systems is biult in correspondence with the following formula:

$$H(2n) = H(n) + H(n)j(n), \tag{1}$$

V. Kumar et al. (Eds.): ICCSA 2003, LNCS 2667, pp. 848–854, 2003.

where n is a dimension of algebra ($n = 2^d$; $d = 0, 1, 2, ...$); $j(n)$ is the new hypercomplex unit which does not belong to algebra $H(n)$. The basis of algebra $H(2n)$ is the sequence of hypercomplex units $1, j(1), j(2), ..., j(n), ..., j(2n)$; the basis of algebra $H(n)$ is : $1, j(1), j(2), ..., j(n-1)$.

A rule for doubling the new hypercomplex units is introduced in correspondence with requirements for associativity and commutability. For example, any number of hypercomplex system corresponding to algebra of a dimension 16 has the following view:

$$x(0) + x(1)j(1) + (x(2) + x(3)j(1))j(2) + ((x(4) + x(5)j(1))+$$
$$(x(6) + x(7)j(1))j(2))j(4) + ((x(8) + x(9)j(1)) + (x(10)+$$
$$x(11)j(j(1))j(2) + ((x(12) + x(13)j(1)) + (x(14)+$$
$$x(15)j(1))j(2))j(4))j(8), \tag{2}$$

where $j(1)^2 = -1$; $j(2)^2 = -1$; $j(4)^2 = -1$; $j(8)^2 = -1$.

If $d = 0$ the algebra corresponds to real number system. If $d = 1$ the algebra corresponds to complex number system. The algebra of domension 4 is the result of doubling complex number system.

The following property for the introduced algebra classes $H(n)$ is based on the isomorphism between complex and real numbers complex modulo [2].

Let a complex module is $m = m(1) + m(2)j$ and its norm is:
$N = m(1)^2 + m(2)^2$; $(m(1), m(2)) = 1$.

Then each whole number of algebra $H(2n)$ is comparable with only one real remainder,which corresponds to only one value of set: $0, 1, 2, ..., N-1$. Proof. Let us consider a hypercomplex number $X(2n)$ of a form similar to (2). Extract the following complex numbers within $X(2n)$:

$X(0) + X(1)j(1)$
$X(2) + X(3)j(1)$
$X(4) + X(5)j(1)$
....
$X(2n-2) + X(2n-1)j(1)$

In correspondence with the above mentioned declaration each of these numbers is comparable with a single remainder out of a range $[0-(N-1)]$ to complex modulus m.

Denote these remainders as $k(1), k(2), k(3), ...$ Now form the following hypercomplex number within algebra $H(n)$

$$K(n) = k(1) + k(2)j(2) + (k(3) + k(4)j(2))j(4) + ... \tag{3}$$

Evidently, (2) and (3) are connected by the congruence relation

$$X(2n) \equiv \langle K(n) \rangle modm \tag{4}$$

The basis of algebra classes H is linearly independent. Becuase of the number $X(2n)$ of algebra $H(2n)$ is comparable with a single remainder $K(n)$, which is a hypercomplex number of algebra $H(n)$.

Thus this procedure sets isomorphism between hypercomplex remainders of algebra $H(2n)$ and $H(n)$ modulo m. Applying it d times we shall obtain the real remainder having the single significance from set: $0, 1, 2, ..., N-1$.

2 Matrix Presentation of Associative-Commutative Algebra

Both Hamilton systems and associative-commutative systems can be presented in matrix form. So a system of dimension 4 has four different methods of a presentation in matrix form. For example, the matrix form of such system has the following view for one of four possible basises:

$$X = x(0)U(4) + x(1)j(14) + x(2)j(24) + x(3)j(34) =$$
$$\begin{pmatrix} x(0) & -x(1) & -x(2) & x(3) \\ x(1) & x(0) & -x(3) & -x(2) \\ x(2) & -x(3) & x(0) & -x(1) \\ x(3) & x(2) & x(1) & x(0) \end{pmatrix}$$

The important significance for further considerations has a conception of conjugate hypercomplex numbers Z and Z' which belong to $H(2n)$ and are determined as

$Z = X + Yj(n)$ and $Z' = X' - Y'j(n)$,

where X, X', Y, Y' belong $H(n)$ and each of couple X, X' and Y, Y' presents conjugate numbers within $H(n)$.

Coefficients of numbers X are elements of the first column of its matrix presentation or elements of the first array of the matrix presentation of number X'. For example, if

$$X = x(0) + x(1)j(14) + x(2)j(24) + x(3)j(34) \tag{5}$$

then

$$X' = x(0) - x(1)j(14) - x(2)j(24) + x(3)j(34) \tag{6}$$

The matrix form of X was presented above and X' has view

$$\begin{pmatrix} x(0) & x(1) & x(2) & x(3) \\ -x(1) & x(0) & -x(3) & x(2) \\ -x(2) & -x(3) & x(0) & x(1) \\ x(3) & -x(2) & -x(1) & x(0) \end{pmatrix}$$

3 Matrix-Vector Procedures

Elements of two arbitrary square matrix-multipliers X, Y are chosen from associative-commutative linear algebra $H(e)$. Let

$$X :=: [x(ij)] \tag{7}$$

and

$$Y :=: [y(ij)], \tag{8}$$

where $i, j = 1, ..., n$.

Let us present the matrix X-premultiplier as a vector-column:

$$X :=: [X'(u)], \tag{9}$$

where $X(u), X'(u)$ belongs to $H(n)$ and are conjugate numbers; $u = 1, 2, ..., n$.

Coefficients of number $X(u)$ are elements of $u - th$ array of matrix X. Here coefficients of number within algebra $H(n)$ are numbers of algebra $H(e)$, that is, they can be real and complex numbers or numbers from algebraic systems obtained by the procedure of doubling. It means that algebra $H(e)$ is embedded in algebra $H(n)$.

Now present the matrix Y-postmultiplier as a vector-array:

$$Y :=: [Y(v)], \tag{10}$$

where $Y(v)$ belong to $H(n)$ and its coefficients are elements of $v - th$ column of the matrix Y; $v = 1, 2, ..., n$.

The basis of hypercomplex system corresponding to algebra $H(n)$ is

$$1, j(1), j(2), ..., j(n-1) \tag{11}$$

Then numbers $X'(u)$ and $Y(v)$ will have the following view within this basis:

$X'(u) = x(u1) + / - x(u2)j(1) + / - ... + / - x(uv)j(v-1) + / - ... + / - x((un)j(n-1)$,

where choosing sign "+" or "-" depends on a singularities of conjugate numbers $X(u)$ and $X'(u)$ ($x(uv)$ is the element of matrix X) and

$Y(v) = y(1v) + y(2v)j(1) + ... + y(uv)j(u-1) + ... + y(nv)j(n-1)$,

where $y(uv)$ is the element of matrix Y.

It was already noticed that the elements of matrices $x(uv)$ and $y(uv)$ belong $H(n)$.

Form a matrix $Z(e)$ from all possible pair products of numbers $X'(u)$ and $Y(v)$.

A part of matrix $Z(e)$

$$\begin{pmatrix} X'(1)Y(1) \; ... \; X'(1)Y(v) \; ... \; X'(1)Y(n) \\ X'(2)Y(1) \; ... \; X'(2)Y(v) \; ... \; X'(2)Y(n) \\ .... \; ... \qquad .... \; ... \qquad .... \\ X'(u)Y(1) \; ... \; X'(u)Y(v) \; ... \; X'(u)Y(n) \\ .... \; ... \qquad .... \; ... \qquad .... \\ X'(n)Y(1) \; ... \; X'(u)Y(v) \; ... \; X'(n)Y(n) \end{pmatrix}$$

belonging to algebra $H(e)$ is equal the product of square matrixes X and Y.

In order to prove this equaty it is enough to show that the part of the product $X'(u)Y(v)$ belonging to algebra $H(e)$ is a significance of the element $z(uv)$ of matrix product $Z = XY$, that is,

$z(uv) = \sum_{i=1}^{n} x(ui)y(iv)$, where $i = 1, ..., n$.

On nother hand, a part of number-product $X'(u)Y(v)$ belonging to algebra $H(e)$ is the sum of corresponding pair products of elements of the first upper array of number $X'(u)$ in its regular matrix presentation and elements of the first left column of number $Y(v)$ of its regular matrix presentation. The first matrix array of number $X'(u)$ corresponds to $u - th$ array of matrix X and the first matrix column corresponds to $v - th$ column of matrix Y.

Thus, the part of the product $X'(u)Y(v)$ belonging to algebra $H(e)$ fully coincides with the element $z(uv)$ of matrix product of matrixes X and Y.

The algorithm of multiplying two matrixes includes the following stages:

1. Direct transforming consisting in

- presenting matrix-premultiplier as the hypercomplex number;
- presenting matrix-postmultiplier as the hypercomplex number;
- transforming the obtained hypercomplex multiplicands into real numbers.

2. Multiplying real numbers.

3. Inverse transforming which includes obtaining number-product of the matrix and determining the real part of this matrix.

Summerizing can be performed by a similar method. Indeed, the sum of numbers of algebra H is the number of this algebra and the matrix obtained from this sum is the sum of matrixes obtained from addends.

4 Versions of Decisions for Computing

It was above mentioned the suggested method includes, as the main element, transforming hypercomplex numbers into real whole numbers and vice versa. As it was shown in the section 1 the both direct and inverse transforming is based on using the property of isomorphism of complex and real numbers [2]. If a norm N for complex modulo $m = m_1 + m_2 j$ is equal to $N = m_1^2 + m_2^2$ and m_1, m_2 are mutually simple numbers then any whole complex number is comparable with one and only one whole number from the set: $0, 1, 2, ..., N - 1$. In other words, if any whole complex number is $c = c_1 + c_2 j$ (c_1 and c_2 are whole numbers) there always is a congruence of a view:

$$c = c_1 + c_2 j \equiv r(mod m), \tag{12}$$

where r is whole number; $r < N$. In this case multiplying/summing two complex numbers is reduced to multiplying/summing of only two whole numbers modulo N. At the same time, a hypercomplex number of any order $2n = 2 * 2^d$ (d=0,1,2,...) and a form similar to (12) can be mapped onto a field of complex numbers and, then, real numbers by above method. This process means consecutive (step-by-step) extracting groups of complex numbers of a view $c_1 + c_2 j(i)$ and their transforming into whole numbers on the basis of (12); i=1,...,d. The last step means obtaining the final single whole number corresponding to the initial hypercomplex number. Thus, the result of operations on hypercomplex numbers presented by real numbers also is real number. The inverse transforming means a reverse process of step-by-step obtaining a group of complex numbers from coefficients of complex numbers obtained within a preceding step. Evidently, the first step means obtaining only one complex number from one initial real number. Consider this process on the following simple example for quaternions. Mapping a quaternion onto a field of complex numbers includes presenting the initial quaternion

$$X = x_0 + x_1 j_1 + x_2 j_2 + x_3 j_1 j_2$$

in the form

$$X = (x_0 + x_1 j_1) + (x_2 + x_3 j_1) j_2 = c_1 + c_2 j_2$$

In beginning, complex numbers c_1 and c_2 are transformed into real numbers r_1 and r_2 respectively. Then, the obtained complex number $r* = r_1 + r_2 j_2$ is transformed into one whole number-remainder r complex modulo m. Now necessary algebraic operations can be executed in field of whole numbers-remainders modulo equal to the norm N of complex modulus m. Naturally, it is rational to use within computers values of moduli equal approximately to a value of a binary computer word.

The inverse transformation means the reverse process of step-by-step obtaining group of complex numbers from whole coefficients of complex numbers obtained in the preceding step. For example, this process for quaternion will be to consist of two steps:

- transforming real number r into a complex number $r*$ $(r* = r_1 + r_2 j_2)$;
- transforming r_1 and r_2 into two complex numbers respectively:

$$c_1 + c_2 j_1$$

and

$$c_3 + c_4 j_1.$$

Thus, the obtained quaternion has the view:

$$c_1 + c_2 j_1 + c_3 j_2 + c_4 j_1 j_2.$$

Evidently, the above described computing methods for matrix-vector procedures are highly efficient if a number of these procedures exeeds significantly a number of isomorphic transforms or if these transforms are performed with separate hardware tools. The most suitable structure of computer system for the second version is multimicroprocessor or massive parallel processor (MPP) consisting of processor elements (PE). All PEs within such MPP performs module algebraic procedures as well as transforming separate matrix arrays/columns (as hypercomplex numbers) into real numbers in parallel. The rational size of MPP is n or $2n$ PEs where $n * n$ is a maximum or middle dimension of matrix operands. $2n$ PEs are necessary for parallel executing transforms on two matrices. A maximum size of MPP is respectively n^2 or $2n^2 PEs$. The method discussed in the paper is efficient if $n > 100$. Evidently, general purpose microprocessors can be used as processor elements within MPP. An implementation of the described computing technology for multimicroprocessor/MPP means fully software version.

5 Conclusion

Efficient computer algebra for matrix-vector processing was presented in this paper. It allows, for example, to reduce multiplying/summerizing two real square large size matrixes to multiplying/summerizing two real numbers-remainders, corresponding to hypercomplex "images" of the matrices. Therefore, a computation complexity of multiplying two matrices presented in hypercomplex systems equals to $O(1) = 1$. In the same time a complexity of multiplying two real matrices by traditional parallel methods oscillates between values $O(n^2)$ and $O(n^3)$ where $n * n$ is a dimension of square matrix operands.

The described above technology is highly efficient if a number of matrix-vector procedures significantly exeeds a number of isomorphic transforms or if these transforms are performed on the basis of multimicroprocessor or other parallel structure, for example, massive parallel processor (MPP). Evidently, general purpose and special microprocessors can be used as processor elements within MPP. All microprocessors can perform algebraic procedures on the basis of concrete moduli as well as transforming separate arrays/columns (as hypercomplex numbers) into real numbers in parallel.

An efficiency of the presented method in a comparison with traditional parallel methods can be determined as a ratio their computation complexities. Taking in account 'expenses' for above modular transforming operations, a value of this ratio oscillates between n^2 and 1. A value of the ratio depends on a volume of transforming operations within problems solved. In the worst case this ratio equals to 1; in the best case the ratio is approximately limited by n^2.

6 Acknowledgment

Authors are very gratitude to NATO, NASA, NSF, and ARO for support of this research.

References

1. W.R. Hamilton, "Optics, Dinamics, Quaternions", Selected proceedings, Moscow, Science, 1994, pp. 345-437.
2. V.M. Vinogradov, "Number Theory", Moscow, 'Science', 1982.